1. Introduction

Models where leader firms make commitments to gain a better market position have been commonly represented in two ways: settings à la Stackelberg and setups with strategic use of investments. In the standard case with one leader and one follower, outcomes are indeterminate and crucially depend on whether prices/quantities display strategic substitutability or complementarity. Thus, firms could end up behaving more aggressively or accommodating entry (Fudenberg and Tirole, 1984).

Etro (2006; 2008) and Anderson et al. (2020) have recently shown that the conclusions of this model differ when there is free entry of followers. Their main insight is that, irrespective of whether prices/quantities are strategic substitutes or complements, a leader always behaves more aggressively and limits entry of followers. In particular, Etro (2006) is the first study to characterize strategic investments under endogenous entry by considering investments that do not directly impact rivals' profits.

In this paper, we characterize an endogenous-entry model à la Etro where demand-enhancing investments additionally affect the competitive environment, and so directly impact rivals' profits. This characterization of investments makes it possible to reflect that, if we compare two identical economies, competition is tougher in the one whose products are more appealing in non-price dimensions.

Our framework allows for an arbitrary number of possibly-heterogeneous leaders. This enables us to accommodate a rich set of scenarios, where not all leaders have the same profitability or importance for aggregate outcomes. Furthermore, we consider that competition is in quantities, in contrast to the analysis in Alfaro and Lander (2020) performed under price competition. The study of the Cournot-competition case is relevant since the type of competition is ultimately industry-dependent and most of the outcomes in our model are indeterminate. Consequently, our findings extend results to markets better characterized by quantity competition.

With the goal of isolating the strategic motives to invest, we follow the traditional approach by Fudenberg and Tirole (1984). This requires comparing the outcomes emerging in two games. In the first one, which we refer to as a sequential-moves game, leaders choose demand-enhancing investments prior to the entry decisions of followers and the market stage. The second one is referred to as a simultaneous-moves game. It constitutes a non-strategic benchmark, where a leader's investment decision is not observed by followers and hence cannot be used strategically. The comparison of both games identifies the strategic use of investments, along with its impact on market outcomes.

Our results indicate that each leader strategically chooses its investment to strengthen competition and restrict entry, allowing each to garner greater profit. Nonetheless, unlike Etro (2006), the specific way to increase competition, along with the rest of the outcomes, is indeterminate. Due to this, we provide conditions in terms of demand primitives to ensure that each leader strengthens competition by product innovating more (i.e., increasing its investments) and to know whether quantities, revenues, and prices increase or decrease by deploying such strategy.

We conclude the analysis by illustrating how the conditions to identify outcomes can be applied to a specific case: a quality-augmented CES demand. Under this demand, our results establish that each leader over-invests, sells more units at a higher price, and increases its revenue.

2. Model Setup

There is an industry comprising a horizontally differentiated good.[1] Each firm produces a unique variety $\omega \in \overline{\Omega}$, where $\overline{\Omega}$ denotes the set of all conceivable varieties. Thus, we refer to a firm or variety indistinctly. Moreover, $\overline{\Omega}$ is partitioned into subsets $\mathscr{L}$ and $\mathcal{F}$, which are mnemonics for "leaders" and "followers", respectively.

Each leader ω has marginal costs c_ω, which we suppose are common knowledge. Also, followers have a symmetric marginal cost $c_{\mathcal{F}}$, where $c_\omega < c_{\mathcal{F}}$ for each $\omega \in \mathscr{L}$, and have to pay a fixed sunk cost F to become active and serve the market.

In any equilibrium considered, we suppose there is a set of followers that are active, which entails that all leaders are active too. The subset of firms serving the market is denoted by Ω, where M denotes the number of followers that are active in the market. We simplify the analysis by assuming that M is a real number, so that a zero-profits condition emerges given free entry of followers.

If firm ω is active, it decides on quantities q_ω and investments z_ω. Furthermore, z_ω entails sunk expenditures $f_z(z_\omega)$, where f_z is convex and satisfies $f_z(0) = 0$. We denote a strategy for ω by $\mathbf{x}_\omega := (q_\omega, z_\omega)$, and a profile of strategies for active firms by $\mathbf{x} := (\mathbf{x}_\omega)_{\omega \in \Omega}$.

As for demand, each firm ω has an inverse-demand function $p(\mathbf{x}_\omega, \mathbb{A})$, where $\frac{\partial p(\mathbf{x}_\omega, \mathbb{A})}{\partial \mathbb{A}} < 0$, $\frac{\partial p(\mathbf{x}_\omega, \mathbb{A})}{\partial q_\omega} < 0$, and $\frac{\partial p(\mathbf{x}_\omega, \mathbb{A})}{\partial z_\omega} > 0$. Following Acemoglu and Jensen (2013), we refer to $\mathbb{A}$ as an aggregate, which corresponds to a value in the range of the aggregator $\mathcal{A}(\mathbf{x}) := H\left[\sum_{\omega' \in \Omega} h(\mathbf{x}_{\omega'})\right]$ that satisfies $H' > 0$, $\frac{\partial h(\mathbf{x}_{\omega'})}{\partial q_{\omega'}} > 0$, and $\frac{\partial h(\mathbf{x}_{\omega'})}{\partial z_{\omega'}} > 0$. Higher values of $\mathbb{A}$ represent tougher competition, since greater quantities and investments increase $\mathbb{A}$, which in turn decreases each firm's demand. Examples of demands satisfying this functional form are augmented versions of the linear and CES inverse demands.

The fact that the demand system depends on a firm's own strategy and a function aggregating the strategies of all firms entails that the game is aggregative in the sense of Cornes and Hartley (2012).[2] In such games, a firm's profits function and its derivatives can be described by its own strategy and a scalar that is a function of all firms' strategies. This allows for a parsimonious way to express equilibrium conditions, which we exploit throughout the paper.

3. Equilibrium

To isolate the strategic motive to invest, we follow the standard approach by Fudenberg and Tirole (1984). This requires comparing the outcomes in two scenarios. We respectively denominate them as a simultaneous-moves and sequential-moves game.

The timing of the sequential-moves game is such that leaders make investment choices at the first stage. Followers observe these decisions and make an entry choice. At the final

[1] Cournot has traditionally been considered more appropriate for commodity-like products, such as agricultural goods (Vives, 2001). Nonetheless, following the discussions in studies such as Hortaçsu and Syverson (2004) and Foster et al. (2008; 2016), seemingly homogeneous goods are actually highly differentiated. In particular, Foster et al. (2016) consider markets with physically homogeneous goods such as boxes, carbon black, ready-mixed concrete, and ice. They show that, even when these goods are commodity-like products, differences in the profitability of firms are due to demand-related features, rather than productivity.

[2] Aggregative games have been recently put forth by Acemoglu and Jensen (2013), Nocke and Schutz (2018), and Anderson et al. (2020). For a survey, see Jensen (2018).

stage, both types of firms compete in the market by choosing quantities, while followers additionally decide on investments. As for the simultaneous-moves game, it constitutes a non-strategic benchmark. Its timing is similar to the sequential-moves game, with the only difference that leaders choose their investments concurrently with quantities at the market stage. By comparing the outcomes in both scenarios, we identify a leader's incentive to modify its investment when a group of firms condition their decisions on it.

In this section, we derive the equilibrium for each of these scenarios. In the next one, we establish the main results of the paper. For each game, we suppose that the equilibrium exists, is unique, and interior. Furthermore, we consider that each leader's profit function evaluated at optimal quantities is strictly quasi-concave in investments.

3.1. Simultaneous-Moves Equilibrium

The simultaneous-moves game comprises two stages. In the first one, each follower decides whether to enter the industry by paying F or remain inactive. After this, leaders and active followers decide on quantities and investments at the market stage. We characterize the equilibrium by utilizing an aggregative-games approach, which requires expressing all the solutions in terms of the aggregate.

A firm ω has gross profits given by

$$\pi_{\omega}\left[\mathbf{x}_{\omega}, \mathcal{A}\left(\mathbf{x}\right)\right] := q_{\omega}\left[p\left(\mathbf{x}_{\omega}, \mathcal{A}\left(\mathbf{x}\right)\right) - c_{\omega}\right] - f_{z}\left(z_{\omega}\right). \tag{1}$$

The first-order conditions that characterize optimal quantities and investments for an active firm ω are

$$\frac{\partial \pi_{\omega}\left(\mathbf{x}_{\omega}, \mathbb{A}\right)}{\partial q_{\omega}} + \frac{\partial \pi_{\omega}\left(\mathbf{x}_{\omega}, \mathbb{A}\right)}{\partial \mathbb{A}} \frac{\partial \mathcal{A}\left(\mathbf{x}\right)}{\partial q_{\omega}} = 0, \tag{2}$$

$$\gamma_{\omega}^{\text{sim}}\left(\mathbf{x}_{\omega}; \mathbb{A}\right) := \frac{\partial \pi_{\omega}\left(\mathbf{x}_{\omega}, \mathbb{A}\right)}{\partial z_{\omega}} + \frac{\partial \pi_{\omega}\left(\mathbf{x}_{\omega}, \mathbb{A}\right)}{\partial \mathbb{A}} \frac{\partial \mathcal{A}\left(\mathbf{x}\right)}{\partial z_{\omega}} = 0. \tag{3}$$

This system determines ω's best-response functions. Alternatively, ω's optimal strategy can be characterized as a function of the aggregate, consistent with an aggregative view of the game. Specifically, we express ω's optimal strategy by $\mathbf{x}_{\omega}\left(\mathbb{A}\right) := \left[q_{\omega}\left(\mathbb{A}\right), z_{\omega}\left(\mathbb{A}\right)\right]$, where $\mathbf{x}_{\mathcal{F}}\left(\mathbb{A}\right)$ denotes the optimal strategy of a firm ω with marginal cost $c_{\mathcal{F}}$.

By expressing optimal strategies in this way, the Nash equilibrium at the market stage requires that the firms' optimal decisions self-generate the value $\mathbb{A}$. Formally,

$$\mathcal{A}^{\text{sim}}\left(\mathbb{A}, M\right) = \mathbb{A}, \tag{NE-sim}$$

where $\mathcal{A}^{\text{sim}}$ corresponds to $\mathcal{A}$ evaluated at the backward-response functions and is given by

$$\mathcal{A}^{\text{sim}}\left(\mathbb{A}, M\right) := H\left\{ M h\left[\mathbf{x}_{\mathcal{F}}\left(\mathbb{A}\right)\right] + \sum_{\omega \in \mathscr{L}} h\left[\mathbf{x}_{\omega}\left(\mathbb{A}\right)\right] \right\}. \tag{4}$$

Furthermore, the zero-profits condition is

$$\pi_{\mathcal{F}}\left(\mathbb{A}\right) = F, \tag{ZP}$$

where $\pi_{\mathcal{F}}\left(\mathbb{A}\right)$ is the optimal profit of a firm with marginal cost $c_{\mathcal{F}}$, i.e. (1) evaluated at $\mathbf{x}_{\mathcal{F}}\left(\mathbb{A}\right)$.

Overall, the aggregative-games approach determines that the equilibrium can be identified through values M^{sim} and $\mathbb{A}^{\text{sim}}$ that satisfy conditions (NE-sim) and (ZP). In particular, once that $\mathbb{A}^{\text{sim}}$ is obtained, the equilibrium decisions of any firm can be determined. They include investments, quantities, and prices.

Additionally, through inspection of (NE-sim) and (ZP), we can appreciate that $\mathbb{A}^{\text{sim}}$ is completely determined by (ZP). This reflects that there is only one equilibrium aggregate that is consistent with zero profits. Thus, interpreting the aggregate as a scalar that captures the level of competition, (ZP) identifies the competitive environment in equilibrium. The relevance of this is that, since the aggregate is a sufficient statistic for a firm's decisions, we do not need to solve for (NE-sim) if our goal is to characterize a leader's equilibrium investment, quantity, or price.

3.2. Sequential-Moves Equilibrium

The timing of the sequential-moves scenario is the same as in the simultaneous-moves case. The only difference is that leaders decide on investments at the beginning of the game.

For the equilibrium characterization of this game, we also follow an aggregative-games approach. Unlike the simultaneous-move game, the solution of the market stage defines a class of subgames for each vector of leaders' investments, $\mathbf{z}^{\mathscr{L}} := (z_\omega)_{\omega\in\mathscr{L}}$. Nonetheless, this does not affect the characterization of optimal decisions. Thus, $\mathbf{x}_{\mathcal{F}}(\mathbb{A})$ is still characterized by (2) and (3) with marginal cost $c_{\mathcal{F}}$. Moreover, leader ω's optimal quantities are still given by (2), which determines a function $q_\omega(z_\omega, \mathbb{A})$.

Given these optimal choices, the condition for a Nash equilibrium at the market stage is akin to that in the simultaneous-moves scenario. Specifically, given $\mathbf{z}^{\mathscr{L}}$, there is a Nash equilibrium at the market stage when $\mathbb{A}$ constitutes a fixed point of $\mathcal{A}^{\text{seq}}$:

$$\mathcal{A}^{\text{seq}}\left(\mathbb{A}, M, \mathbf{z}^{\mathscr{L}}\right) = \mathbb{A}, \tag{NE-seq}$$

where

$$\mathcal{A}^{\text{seq}}\left(\mathbb{A}, M, \mathbf{z}^{\mathscr{L}}\right) := Mh\left[\mathbf{x}_{\mathcal{F}}(\mathbb{A})\right] + \sum_{\omega\in\mathscr{L}} h\left[q_\omega(z_\omega, \mathbb{A}), z_\omega\right]. \tag{5}$$

As for free entry, the zero-profits condition is still given by (ZP). Therefore, the same equilibrium aggregate holds in each scenario, $\mathbb{A}^{\text{seq}} = \mathbb{A}^{\text{sim}}$, which we refer to as $\mathbb{A}^*$. Intuitively, it reflects that any variation in z_ω by leader ω triggers changes in M such that the aggregate does not vary.

Given $\mathbb{A}^*$, optimal investments of leader ω are determined as the solution that maximizes $\pi_\omega\left[q_\omega(z_\omega; \mathbb{A}^*), z_\omega; \mathbb{A}^*\right]$. Therefore, they can be characterized by the first-order condition of this problem, which is

$$\gamma_\omega^{\text{seq}}(\mathbf{x}_\omega; \mathbb{A}) := \frac{\partial \pi_\omega\left[q_\omega(z_\omega; \mathbb{A}^*), z_\omega; \mathbb{A}^*\right]}{\partial z_\omega} - \frac{\partial \pi_\omega\left[q_\omega(z_\omega; \mathbb{A}^*), z_\omega; \mathbb{A}^*\right]}{\partial q_\omega}\frac{\partial q_\omega(z_\omega, \mathbb{A}^*)}{\partial z_\omega} = 0. \tag{6}$$

4. Results

The following proposition states how leaders strategically vary their investment choices when followers condition on them. This is done by comparing the equilibrium in the simultaneous-moves and sequential-moves games. Proofs of all the propositions are relegated to the appendix.

Proposition 4.1. *Relative to the simultaneous-moves equilibrium, in the sequential-moves equilibrium each leader varies its investment to strengthen competition. Additionally, each leaders increases its profit and the number of followers decreases.*

The fact that leader ω strengthens competition means that ω chooses its investments to increase the term $h(\mathbf{x}_\omega)$. Nonetheless, the strategy to increase h is setup-specific and gives rise to various possible outcomes regarding the rest of its variables. Thus, next, we establish assumptions in terms of demand primitives to ensure that leaders product innovate more (i.e., over-invest). Conditional on over-investing, we also identify conditions to know whether each leader increases or decreases its revenue, quantity, and price.

With this goal, let $\varepsilon^q(\mathbf{x}_\omega, \mathbb{A}) := -\frac{\mathrm{d}\ln p_\omega(\mathbf{x}_\omega,\mathbb{A})}{\mathrm{d}\ln q_\omega}$ be the quantity elasticity of the inverse demand. Moreover, let $\xi^q(\mathbf{x}_\omega, \mathbb{A}) := -\frac{\partial \ln p_\omega(\mathbf{x}_\omega,\mathbb{A})}{\partial \ln q_\omega}$ and $\xi^z(\mathbf{x}_\omega, \mathbb{A}) := -\frac{\partial \ln p_\omega(\mathbf{x}_\omega,\mathbb{A})}{\partial \ln z_\omega}$ be the quantity and investments elasticity when the impact on $\mathbb{A}$ is ignored, respectively. Finally, denote

$$\lambda(\mathbf{x}_\omega, \mathbb{A}) := \frac{\frac{1-\varepsilon^q(\mathbf{x}_\omega,\mathbb{A})}{\varepsilon^q(\mathbf{x}_\omega,\mathbb{A})}\xi^z(\mathbf{x}_\omega, \mathbb{A}) - \frac{\partial \ln \varepsilon^q(\mathbf{x}_\omega;\mathbb{A})}{\partial \ln z_\omega}}{1-\varepsilon^q(\mathbf{x}_\omega, \mathbb{A}) + \frac{\partial \ln \varepsilon^q(\mathbf{x}_\omega;\mathbb{A})}{\partial \ln q_\omega}},$$

whose relevance comes from that $\frac{\partial \ln q_\omega(z_\omega;\mathbb{A})}{\partial \ln z_\omega} = \lambda[q_\omega(z_\omega;\mathbb{A}), z_\omega, \mathbb{A}]$.

Proposition 4.2. *Relative to the simultaneous-moves equilibrium, in the sequential-moves equilibrium each leader ω over-invests if $\lambda(\mathbf{x}_\omega, \mathbb{A}) > -\frac{\partial \ln h(\mathbf{x}_\omega)}{\partial \ln z_\omega}\left(\frac{\partial \ln h(\mathbf{x}_\omega)}{\partial \ln q_\omega}\right)^{-1}$. If additionally $\lambda(\mathbf{x}_\omega, \mathbb{A}) > \frac{\xi^z(\mathbf{x}_\omega,\mathbb{A})}{\xi^q(\mathbf{x}_\omega,\mathbb{A})-1}$ then ω increases its revenues, and if the inequality is reversed it decreases its revenues. If additionally $\lambda(\mathbf{x}_\omega, \mathbb{A}) < \frac{\xi^z(\mathbf{x}_\omega,\mathbb{A})}{\xi^q(\mathbf{x}_\omega,\mathbb{A})}$ then ω increases its prices, and if the inequality is reversed it lowers its prices. If additionally the numerator of λ is positive then ω increases its quantities, and otherwise it lowers its quantities.*

The implications of these propositions can be illustrated through a quality-augmented CES inverse demand for variety ω. This is given by

$$p(\mathbf{x}_\omega, \mathbb{Q}) := E(q_\omega)^{\rho-1}(z_\omega)^\delta \mathbb{Q}^{-\rho}, \tag{7}$$

where $0 < \rho < 1$, $0 < \delta < 1$, $\mathbb{Q} := \left[\sum_{\omega\in\Omega}(q_\omega)^\rho (z_\omega)^\delta\right]^{\frac{1}{\rho}}$ is the quantity index, and $E > 0$ is the industry expenditure. For this demand, our characterization of outcomes implies the following.

Corollary 4.3. *Suppose the quality-augmented CES inverse demand given in* (7)*. Relative to the simultaneous-moves equilibrium, in the sequential-move equilibrium each leader strengthens competition and garners greater profits, whereas the number of followers is lower. Additionally, each leader over-invests, increases its revenue, sells greater quantities, and charges a higher price.*

In words, this determines that all the results for a quality-augmented CES inverse demand are determined: leaders product innovate more, increase their revenues, and sell more units at a higher price.

5. Conclusion

We have analyzed an endogenous-entry model under Cournot competition, with leaders strategically investing to gain a better market position. Unlike previous studies assuming free entry of followers, our framework considers multiple heterogeneous leaders and demand-enhancing investments that directly affect rivals' demands.

With the goal of isolating the strategic motives to invest, we have followed the standard approach by Fudenberg and Tirole (1984) and compared the outcomes in two games.

We have referred to the first one as a sequential-moves game, where leaders choose investments prior to each follower's entry decision and the market stage. Due to this, followers make choices condition on each leader's investment. After this, we have derived the equilibrium in a simultaneous-move game, which constitutes a non-strategic benchmark where investments are not observed by followers.

Our results establish that leaders always strengthen competition, thereby restricting entry of followers and allowing leaders to garner greater profits. Nonetheless, the rest of the outcomes are indeterminate. Thus, we have stated conditions in terms of demand primitives to identify when leaders innovate more, and whether this increases or decreases the revenue, quantity, and price of each leader.

We have concluded the analysis by applying the results to the case of a quality-augmented CES inverse demand. Under this demand, we have established that leaders always over-invest, sell more units at a higher price, and increase their revenues.

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Appendix

This appendix contains all the proofs of the paper. Regarding notation, we denote any variable in the equilibrium of the simultaneous- and sequential-moves game with a superscript "sim" and "seq", respectively. For instance, z_ω^{sim} z_ω^{seq} refers to the investments of leader ω in each equilibrium.

Lemma 1. *Let* $q_\omega^* := q_\omega(z_\omega, \mathbb{A})$. *Then,*

$$sgn\left\{\frac{\partial \ln q_\omega(z_\omega;\mathbb{A})}{\partial \ln z_\omega} + \frac{\partial \ln h(q_\omega^*, z_\omega)}{\partial \ln z_\omega}\left(\frac{\partial \ln h(q_\omega^*, z_\omega)}{\partial \ln q_\omega}\right)^{-1}\right\} = sgn\left\{\frac{\mathrm{d}\ln h\left[q_\omega(z_\omega;\mathbb{A}), z_\omega\right]}{\mathrm{d}\ln z_\omega}\right\} = sgn\left\{\frac{\mathrm{d}\ln \mathcal{A}\left[q_\omega(z_\omega;\mathbb{A}), z_\omega; \mathbf{z}_{-\omega}^{\mathscr{L}}, \mathbb{A}\right]}{\mathrm{d}\ln z_\omega}\right\}.$$

Proof of Lemma 1. By definition,

$$\frac{\mathrm{d}\ln h\left[q_\omega(z_\omega;\mathbb{A}), z_\omega\right]}{\mathrm{d}\ln z_\omega} = \frac{\partial \ln h(q_\omega^*, z_\omega)}{\partial \ln q_\omega}\frac{\partial \ln q_\omega(z_\omega;\mathbb{A})}{\partial \ln z_\omega} + \frac{\partial \ln h(q_\omega^*, z_\omega)}{\partial \ln z_\omega}.$$

This implies that $\frac{\mathrm{d}\ln h[q_\omega(z_\omega;\mathbb{A}),z_\omega]}{\mathrm{d}\ln z_\omega} > 0$ iff $\frac{\partial \ln q_\omega(z_\omega;\mathbb{A})}{\partial \ln z_\omega} > -\frac{\partial \ln h(q^*_\omega,z_\omega)}{\partial \ln z_\omega}\left(\frac{\partial \ln h(q^*_\omega,z_\omega)}{\partial \ln q_\omega}\right)^{-1}$, where we have used that $\frac{\partial \ln h(q^*_\omega,z_\omega)}{\partial \ln q_\omega} > 0$. Besides, by using that $\frac{\partial \ln \mathcal{A}(\mathbf{x})}{\partial \ln z_\omega} = \frac{H'\left[H^{-1}(\mathbb{A})\right]}{\mathbb{A}}\frac{\partial h(\mathbf{x}_\omega)}{\partial \ln z_\omega}$ and $\frac{\partial \ln \mathcal{A}(\mathbf{x})}{\partial \ln q_\omega} = \frac{H'\left[H^{-1}(\mathbb{A})\right]}{\mathbb{A}}\frac{\partial h(\mathbf{x}_\omega)}{\partial \ln q_\omega}$, then

$$\frac{\mathrm{d}\ln \mathcal{A}\left[q_\omega\left(z_\omega;\mathbb{A}\right),z_\omega;\mathbf{z}^{\mathscr{L}}_{-\omega},\mathbb{A}\right]}{\mathrm{d}\ln z_\omega} = \frac{H'\left[H^{-1}\left(\mathbb{A}\right)\right]}{\mathbb{A}}\left[\frac{\mathrm{d}\ln h\left[q_\omega\left(z_\omega;\mathbb{A}\right),z_\omega\right]}{\mathrm{d}\ln z_\omega}\right].$$

Since $H' > 0$, the result follows. ■

Lemma 2. *Let $q^{sim}_\omega := q_\omega\left(z^{sim}_\omega;\mathbb{A}^*\right)$ where $\mathbb{A}^*$ is the equilibrium aggregate in both scenarios.* ***Case i)*** *If $\frac{\partial \ln q_\omega\left(z^{sim}_\omega;\mathbb{A}^*\right)}{\partial \ln z_\omega} < -\frac{\partial \ln h\left[q_\omega\left(z^{sim}_\omega;\mathbb{A}^*\right),z^{sim}_\omega\right]}{\partial \ln z_\omega}\left(\frac{\partial \ln h\left(q^{sim}_\omega,z^{sim}_\omega\right)}{\partial \ln q_\omega}\right)^{-1}$, then $z^{sim}_\omega > z^{seq}_\omega$.* ***Case ii)*** *If $\frac{\partial \ln q_\omega\left(z^{sim}_\omega;\mathbb{A}^*\right)}{\partial \ln z_\omega} > -\frac{\partial \ln h\left[q_\omega\left(z^{sim}_\omega;\mathbb{A}^*\right),z^{sim}_\omega\right]}{\partial \ln z_\omega}\left(\frac{\partial \ln h\left(q^{sim}_\omega,z^{sim}_\omega\right)}{\partial \ln q_\omega}\right)^{-1}$, then $z^{seq}_\omega > z^{sim}_\omega$.*

Proof of Lemma 2. Consider leader ω. The marginal profits of investments in the simultaneous and sequential case are respectively $\gamma^{\text{sim}}_\omega\left(\mathbf{x}_\omega;\mathbb{A}\right)$ and $\gamma^{\text{seq}}_\omega\left(\mathbf{x}_\omega;\mathbb{A}\right)$, as defined in (3) and (6). Using the characterization of optimal quantities given by $\frac{\partial \pi_\omega(\mathbf{x}_\omega,\mathbb{A})}{\partial q_\omega} = -\frac{\partial \pi_\omega(\mathbf{x}_\omega,\mathbb{A})}{\partial \mathbb{A}}\frac{\partial \mathcal{A}(\mathbf{x}_\omega,\mathbb{A})}{\partial q_\omega}$, we can reexpress $\gamma^{\text{seq}}_\omega\left(\mathbf{x}_\omega;\mathbb{A}\right) := \frac{\partial \pi_\omega(\mathbf{x}_\omega,\mathbb{A})}{\partial z_\omega} - \frac{\partial \pi_\omega(\mathbf{x}_\omega,\mathbb{A})}{\partial \mathbb{A}}\frac{\partial \mathcal{A}(\mathbf{x}_\omega,\mathbb{A})}{\partial q_\omega}\frac{\partial q_\omega(z_\omega;\mathbb{A})}{\partial z_\omega}$. Let $\Delta_\omega\left(z_\omega;\mathbb{A}^*\right) := \gamma^{\text{seq}}_\omega\left[q_\omega\left(z_\omega;\mathbb{A}^*\right),z_\omega;\mathbb{A}^*\right] - \gamma^{\text{sim}}_\omega\left[q_\omega\left(z_\omega;\mathbb{A}^*\right),z_\omega;\mathbb{A}^*\right]$, which is equivalent to

$$\Delta_\omega\left(z_\omega;\mathbb{A}^*\right) = -\frac{\partial \pi_\omega\left(q^*_\omega,z_\omega,\mathbb{A}^*\right)}{\partial \mathbb{A}}\left[\frac{\partial \mathcal{A}\left(q^*_\omega,z_\omega;\mathbf{z}^{\mathscr{L}}_{-\omega},\mathbb{A}^*\right)}{\partial q_\omega}\frac{\partial q_\omega\left(z_\omega;\mathbb{A}^*\right)}{\partial z_\omega} + \frac{\partial \mathcal{A}\left(q^*_\omega,z_\omega;\mathbf{z}^{\mathscr{L}}_{-\omega},\mathbb{A}^*\right)}{\partial z_\omega}\right],$$

where $q^*_\omega := q_\omega\left(z_\omega;\mathbb{A}^*\right)$. Since $\frac{\partial \pi_\omega(\cdot)}{\partial \mathbb{A}} < 0$, we have that for each $\left(z_\omega,\mathbb{A}^*\right)$,

$$\operatorname{sgn}\left\{\Delta_\omega\left(z_\omega;\mathbb{A}^*\right)\right\} = \operatorname{sgn}\left\{\frac{\mathrm{d}\mathcal{A}\left[q_\omega\left(z_\omega;\mathbb{A}^*\right),z_\omega;\mathbf{z}^{\mathscr{L}}_{-\omega},\mathbb{A}^*\right]}{\mathrm{d}z_\omega}\right\} = \operatorname{sgn}\left\{\frac{\mathrm{d}h\left[q_\omega\left(z_\omega;\mathbb{A}^*\right),z_\omega\right]}{\mathrm{d}z_\omega}\right\}, \tag{8}$$

where the second equality follows by Lemma 1.

Define $\Delta^{\text{sim}}_\omega := \Delta_\omega\left(z^{\text{sim}}_\omega;\mathbb{A}^*\right)$. The strict quasi-concavity of profits evaluated at optimal quantities determines that if we can show that $\Delta^{\text{sim}}_\omega > 0$ then leader ω over-invests, while if $\Delta^{\text{sim}}_\omega < 0$ then it under-invests. Next, we show this result can be applied to the cases considered in the information of the lemma. Case i) implies that $\frac{\mathrm{d}\ln h\left[q_\omega\left(z^{\text{sim}}_\omega;\mathbb{A}^*\right),z^{\text{sim}}_\omega\right]}{\mathrm{d}\ln z_\omega} < 0$, so that, by Lemma 1 and (8), $\Delta^{\text{sim}}_\omega < 0$ and, so, $z^{\text{sim}}_\omega > z^{\text{seq}}_\omega$. Case ii) implies that $\frac{\mathrm{d}\ln h\left[q_\omega\left(z^{\text{sim}}_\omega;\mathbb{A}^*\right),z^{\text{sim}}_\omega\right]}{\mathrm{d}\ln z_\omega} > 0$, which determines that $\Delta^{\text{sim}}_\omega > 0$ and, so, $z^{\text{seq}}_\omega > z^{\text{sim}}_\omega$. ■

Proof of Proposition 4.1. As for profits, the result follows by a revealed-preference argument. This is because the same aggregate holds under the simultaneous and sequential equilibrium. Thus, leader ω can obtain at least the same profits as in the simultaneous scenario by choosing z^{sim}_ω.

To show that leader ω behaves more aggressively, we need to prove that $\Delta h_\omega > 0$ where

$$\Delta h_\omega := h\left[q_\omega\left(z^{\text{seq}}_\omega;\mathbb{A}^*\right),z^{\text{seq}}_\omega\right] - h\left[q_\omega\left(z^{\text{sim}}_\omega;\mathbb{A}^*\right),z^{\text{sim}}_\omega\right].$$

To do this, define Case i) and Case ii) as in Lemma 2. By (8), we get $\Delta h_\omega = \int_{z^{\text{sim}}_\omega}^{z^{\text{seq}}_\omega}\Delta_\omega\left(z;\mathbb{A}^*\right)\mathrm{d}z$. For Case i), we have proved that $\Delta^{\text{sim}}_\omega < 0$ and, so, $z^{\text{sim}}_\omega > z^{\text{seq}}_\omega$.

Moreover, $\Delta_\omega(z) < 0$ for any $z \in \left(z_\omega^{\text{seq}}, z_\omega^{\text{sim}}\right)$. This follows because $\gamma_\omega^{\text{sim}}\left(z_\omega^{\text{sim}}, \mathbb{A}^*\right) = 0$, $\gamma_\omega^{\text{seq}}\left(z_\omega^{\text{seq}}, \mathbb{A}^*\right) = 0$, and $\gamma_\omega^{\text{seq}}\left(z_\omega^{\text{sim}}, \mathbb{A}^*\right) < 0$. In addition, by the strict quasi-concavity of profits and given z such that $z_\omega^{\text{sim}} > z > z_\omega^{\text{seq}}$, then $\gamma_\omega^{\text{seq}}(z;\mathbb{A}^*) < 0$ and $\gamma_\omega^{\text{sim}}(z;\mathbb{A}^*) > 0$. This implies that $\Delta_\omega(z;\mathbb{A}^*) < 0$ for any $z \in \left(z_\omega^{\text{seq}}, z_\omega^{\text{sim}}\right)$. Thus, $\Delta h_\omega > 0$. As for Case ii), we have shown that $\Delta_\omega^{\text{sim}} > 0$ and, so, $z_\omega^{\text{seq}} > z_\omega^{\text{sim}}$. Furthermore, $\gamma_\omega^{\text{sim}}\left(z_\omega^{\text{sim}}, \mathbb{A}^*\right) = 0$, $\gamma_\omega^{\text{seq}}\left(z_\omega^{\text{seq}}, \mathbb{A}^*\right) = 0$, and $\gamma_\omega^{\text{seq}}\left(z_\omega^{\text{sim}}, \mathbb{A}^*\right) > 0$. Additionally, by the strict quasi-concavity of profits and given z such that $z_\omega^{\text{seq}} > z > z_\omega^{\text{sim}}$, then $\gamma_\omega^{\text{seq}}(z;\mathbb{A}^*) > 0$ and $\gamma_\omega^{\text{sim}}(z;\mathbb{A}^*) < 0$. Therefore, $\Delta_\omega(z) > 0$ for any $z \in \left(z_\omega^{\text{sim}}, z_\omega^{\text{seq}}\right)$ and, since, $z_\omega^{\text{seq}} > z_\omega^{\text{sim}}$ then $\Delta h_\omega > 0$.

As for the number of followers, M^{sim} and M^{seq} correspond to the solution to (4) and (5) for a given $\mathbb{A}^*$, respectively. Thus, (NE-sim) and (NE-seq) imply:

$$M^{\text{sim}} h\left[\mathbf{x}_{\mathcal{F}}(\mathbb{A}^*)\right] + \sum_{\omega \in \mathscr{L}} h\left[p_\omega\left(z_\omega^{\text{sim}}; \mathbb{A}^*\right), z_\omega^{\text{sim}}\right] = M^{\text{seq}} h\left[\mathbf{x}_{\mathcal{F}}(\mathbb{A}^*)\right] + \sum_{\omega \in \mathscr{L}} h\left[p_\omega\left(z_\omega^{\text{seq}}; \mathbb{A}^*\right), z_\omega^{\text{seq}}\right],$$

and, so, $M^{\text{seq}} - M^{\text{sim}} = -\frac{\sum_{\omega \in \mathscr{L}} \Delta h_\omega}{h[\mathbf{x}_{\mathcal{F}}(\mathbb{A}^*)]}$. Since we have shown that $\Delta h_\omega > 0$, then $M^{\text{seq}} < M^{\text{sim}}$.■

Lemma 3. *Let* $q_\omega^* := q_\omega(z_\omega, \mathbb{A})$. *Then,*

$$\frac{\partial \ln q_\omega(z_\omega;\mathbb{A})}{\partial \ln z_\omega} = \frac{\frac{1-\varepsilon^q[q_\omega(z_\omega;\mathbb{A}),z_\omega;\mathbb{A}]}{\varepsilon^q[q_\omega(z_\omega;\mathbb{A}),z_\omega;\mathbb{A}]}\xi^z\left[q_\omega(z_\omega;\mathbb{A}), z_\omega;\mathbb{A}\right] - \frac{\partial \ln \varepsilon^q[q_\omega(z_\omega;\mathbb{A}),z_\omega;\mathbb{A}]}{\partial \ln z_\omega}}{1 - \varepsilon^q\left[q_\omega(z_\omega;\mathbb{A}), z_\omega;\mathbb{A}\right] + \frac{\partial \ln \varepsilon^q[q_\omega(z_\omega;\mathbb{A}),z_\omega;\mathbb{A}]}{\partial \ln q_\omega}}, \tag{9}$$

$$sgn\left\{\frac{\partial \ln q_\omega(z_\omega;\mathbb{A})}{\partial \ln z_\omega}\right\} = sgn\left\{\frac{1-\varepsilon^q\left[q_\omega(z_\omega;\mathbb{A}), z_\omega;\mathbb{A}\right]}{\varepsilon^q\left[q_\omega(z_\omega;\mathbb{A}), z_\omega;\mathbb{A}\right]}\xi^z\left[q_\omega(z_\omega;\mathbb{A}), z_\omega;\mathbb{A}\right] - \frac{\partial \ln \varepsilon^q\left[q_\omega(z_\omega;\mathbb{A}), z_\omega;\mathbb{A}\right]}{\partial \ln z_\omega}\right\}. \tag{10}$$

Proof of Lemma 3. The first-order condition for quantities determines that $\frac{p_\omega(\mathbf{x}_\omega,\mathbb{A})}{p(\mathbf{x}_\omega,\mathbb{A})-c_\omega} = \frac{1}{\varepsilon^q(\mathbf{x}_\omega,\mathbb{A})}$. Reexpressing this, it is determined that $\ln c_\omega = \ln\left[1 - \varepsilon^q(\mathbf{x}_\omega,\mathbb{A})\right] + \ln p_\omega(\mathbf{x}_\omega,\mathbb{A})$. To streamline notation, denote $\varepsilon^q(\mathbf{x}_\omega,\mathbb{A})$ by ε_ω^q. Differentiating the expression for a given $\mathbb{A}$, we obtain

$$\left(\frac{-\frac{\partial \ln \varepsilon_\omega^q}{\partial \ln z_\omega}\varepsilon_\omega^q}{1-\varepsilon_\omega^q} + \xi^z\right)\mathrm{d}\ln z_\omega + \varepsilon_\omega^q\left(\frac{-\frac{\partial \ln \varepsilon_\omega^q}{\partial \ln q_\omega}}{1-\varepsilon_\omega^q} - 1\right)\mathrm{d}\ln q_\omega = 0,$$

which, rearranging the expression, gives (9).

Next, we show that (10) holds by establishing that the marginal profits of quantities are increasing in z_ω for a given $\mathbb{A}$. Gross optimal profits of leader ω are given by (1), so that by differentiating it when $\mathcal{A}$ is affected by changes in q_ω,

$$D_q\pi_\omega(\mathbf{x}_\omega,\mathbb{A}) :=: \frac{\mathrm{d}\pi_\omega(\mathbf{x}_\omega,\mathbb{A})}{\mathrm{d}\ln q_\omega} := \kappa(\mathbf{x}_\omega,\mathbb{A})\left[1 - p(\mathbf{x}_\omega,\mathbb{A})\frac{\varepsilon^q(\mathbf{x}_\omega,\mathbb{A})}{p(\mathbf{x}_\omega,\mathbb{A}) - c_\omega}\right],$$

where $\kappa(\mathbf{x}_\omega,\mathbb{A}) := q_\omega\left[p(\mathbf{x}_\omega,\mathbb{A}) - c_\omega\right]$ and satisfies $\kappa > 0$ along the relevant range where $p(\mathbf{x}_\omega,\mathbb{A}) > c_\omega$. Now, we incorporate that the aggregate is unaffected by z_ω. For the range of optimal quantities and after some algebraic manipulation, it is determined that

$$\text{sgn}\left\{\frac{\partial D_q\pi_\omega(\mathbf{x}_\omega;\mathbb{A})}{\partial \ln z_\omega}\right\} = \text{sgn}\left\{\xi^z(\mathbf{x}_\omega;\mathbb{A})\left[1 - \varepsilon^q(\mathbf{x}_\omega;\mathbb{A})\right] - \varepsilon^q(\mathbf{x}_\omega;\mathbb{A})\frac{\partial \ln \varepsilon^q(\mathbf{x}_\omega;\mathbb{A})}{\partial \ln z_\omega}\right\},$$

and the result follows. ■

Proof of Proposition 4.2. Using Lemma 3, notice that $\frac{\partial \ln q_\omega(z_\omega;\mathbb{A})}{\partial \ln z_\omega} = \lambda\left[q_\omega(z_\omega;\mathbb{A}), z_\omega, \mathbb{A}\right]$ by (9). Thus, the assumption $\lambda(\mathbf{x}_\omega,\mathbb{A}) > -\frac{\partial \ln h(\mathbf{x}_\omega)}{\partial \ln z_\omega}\left(\frac{\partial \ln h(\mathbf{x}_\omega)}{\partial \ln q_\omega}\right)^{-1}$ ensures that Case ii) always holds.

Regarding quantities, let $q_\omega^{\text{sim}} := q_\omega\left(z_\omega^{\text{sim}}, \mathbb{A}^*\right)$ and $q_\omega^{\text{seq}} := q_\omega\left(z_\omega^{\text{seq}}, \mathbb{A}^*\right)$. Using optimal quantities, $q_\omega(z_\omega;\mathbb{A})$, then $q_\omega(z_\omega^{\text{seq}};\mathbb{A}^*)-q_\omega\left(z_\omega^{\text{sim}};\mathbb{A}^*\right)=\int_{z_\omega^{\text{sim}}}^{z_\omega^{\text{seq}}}\frac{\mathrm{d}q_\omega(z;\mathbb{A}^*)}{\mathrm{d}z}\mathrm{d}z$. Using that the right-hand side of (10) corresponds to the numerator of λ evaluated at $(q_\omega(z_\omega;\mathbb{A}),z_\omega,\mathbb{A})$, then the result follows.

Furthermore, define leader ω's revenue by $R_\omega := p_\omega q_\omega$ which, given the equilibrium aggregate $\mathbb{A}^*$, can be expressed as a function of its investment: $R_\omega(z_\omega;\mathbb{A}^*):=q\left[p_\omega(z_\omega;\mathbb{A}^*),z_\omega;\mathbb{A}^*\right]p_\omega(z_\omega;\mathbb{A}^*)$. Therefore, $R_\omega(z_\omega^{\text{seq}};\mathbb{A}^*)-R_\omega\left(z_\omega^{\text{sim}};\mathbb{A}^*\right)=\int_{z_\omega^{\text{sim}}}^{z_\omega^{\text{seq}}}\frac{\mathrm{d}R_\omega(z;\mathbb{A}^*)}{\mathrm{d}z}\mathrm{d}z$. Next, we show that $\frac{\mathrm{d}R_\omega(z;\mathbb{A}^*)}{\mathrm{d}z}>0$ for $z\in\left(z_\omega^{\text{sim}},z_\omega^{\text{seq}}\right)$. By definition, $\frac{\mathrm{d}\ln R_\omega(z_\omega;\mathbb{A}^*)}{\mathrm{d}\ln z_\omega}=\xi^z(q_\omega^*,z_\omega;\mathbb{A}^*)-\left[\xi^q(q_\omega^*,z_\omega;\mathbb{A}^*)-1\right]\frac{\partial\ln q_\omega(z_\omega;\mathbb{A}^*)}{\partial\ln z_\omega}$,where $q_\omega^*:=q_\omega(z_\omega;\mathbb{A}^*)$. This implies that $\frac{\mathrm{d}\ln R_\omega(z_\omega;\mathbb{A}^*)}{\mathrm{d}\ln z_\omega}>0$ iff $\frac{\partial\ln q_\omega(z_\omega;\mathbb{A}^*)}{\partial\ln z_\omega}>\frac{\xi^z(q_\omega^*,z_\omega;\mathbb{A}^*)}{\xi^q(q_\omega^*,z_\omega;\mathbb{A}^*)-1}$, which we assume.

Regarding prices, we proceed in a similar fashion to the proof for revenues, so that $\frac{\mathrm{d}\ln p_\omega(z_\omega;\mathbb{A}^*)}{\mathrm{d}\ln z_\omega}>0$ iff $\frac{\partial\ln q_\omega(z_\omega;\mathbb{A}^*)}{\partial\ln z_\omega}<\frac{\xi^z(q_\omega^*,z_\omega;\mathbb{A}^*)}{\xi^q(q_\omega^*,z_\omega;\mathbb{A}^*)}$, which we assume.■

Proof of Corollary 4.3. The results regarding aggressive behavior, profits, and number of followers are determined by Proposition 4.1. Denote $\varepsilon^q(\mathbf{x}_\omega,\mathbb{A})$ by ε^q_ω. Then, given (7), we obtain $\varepsilon^q_\omega := 1-\rho(1-s_\omega)$ where $s_\omega:=\frac{p_\omega q_\omega}{E}$ and it is given by a function $s(\mathbf{x}_\omega,\mathbb{Q})=\frac{(q_\omega)^\rho(z_\omega)^\delta}{\mathbb{Q}^\rho}$. Moreover, $\lambda(\mathbf{x}_\omega,\mathbb{Q})=\frac{\delta(1-s_\omega)}{\varepsilon^q_\omega+\rho s_\omega}>0$. Over-investment follows by using that $-\frac{\partial\ln h(\mathbf{x}_\omega)}{\partial\ln z_\omega}\left(\frac{\partial\ln h(\mathbf{x}_\omega)}{\partial\ln q_\omega}\right)^{-1}=\frac{\delta}{\rho}$ and that $\frac{\delta(1-s_\omega)}{\varepsilon^q_\omega+\rho s_\omega}>\frac{\delta}{\rho}$. Furthermore, the result for revenues follows since $\frac{\xi^z(\mathbf{x}_\omega,\mathbb{A})}{\xi^q(\mathbf{x}_\omega,\mathbb{A})-1}=\frac{\delta}{-\rho}<0$ while $\frac{\delta(1-s_\omega)}{\varepsilon^q_\omega+\rho s_\omega}>0$, so that $\frac{\delta(1-s_\omega)}{\varepsilon^q_\omega+\rho s_\omega}>\frac{\delta}{-\rho}$. The fact that $\frac{\delta(1-s_\omega)}{\varepsilon^q_\omega+\rho s_\omega}>0$ also proves the result about quantities. Finally, regarding prices, it can be proven that $\frac{\delta(1-s_\omega)}{\varepsilon^q_\omega+\rho s_\omega}<\frac{\delta}{1-\rho}=\frac{\xi^z(\mathbf{x}_\omega,\mathbb{A})}{\xi^q(\mathbf{x}_\omega,\mathbb{A})}$.■