International Trade[1]

Lecture Note 3: Heckscher-Ohlin Modal

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[1]The notes are still preliminary and in beta. Please, if you find any typo or mistake, send it to malfaro@ualberta.ca.

Contents

Notation

This is a derivation

This is some comment

This is a comment on advanced topics that are not part of the course (you can ignore it without loss of continuity regarding the text)

- The symbol ":=" means "by definition".
- Vectors are denoted by bold lowercase letters (for instance, $\mathbf{x}$) and matrices by bold capital letters (for instance, $\mathbf{X}$).
- The set of nonnegative real numbers is denoted by $\mathbb{R}_+ := [0, \infty)$
- The set of of positive real numbers is denoted by $\mathbb{R}_{++} := (0, \infty)$
- The Cartesian product is denoted by $X_1 \times X_2 \times ... \times X_N$. If each set comprises the nonnegative real numbers, we use the notation $\mathbb{R}_+^N$.
- To differentiate between the verb "maximize" and the operator "maximum", I denote the former with "max" and the latter with "sup" (i.e., supremum). The same caveat applies to "minimize" and "minimum", where I use "min" and "inf", with the latter indicating infimum.
- "iff" means "if and only if"
- $\exp(x)$ is the function e^x.
- Random variables are denoted with a bar below. For instance, $\underline{x}$.

These notes contain hyperlinks in blue and red text. If you are using Adobe Acrobat Reader, you can click on the link and easily navigate back by pressing Alt+Left Arrow.

1 Introduction

In this note, we continue our study of Neoclassical Models. Recall that these models explain trade by considering countries that are inherently different. In particular, the Ricardian model considers that countries differ by their productivity. However, this is not the only source of differences possible, and other sources of heterogeneity have been proposed.

Next, we focus in particular on the Heckscher-Ohlin (HO) model. This explains trade by differences in each country's endowments of factors. The aim of this note is to derive the "four core propositions" of the model:

[1] Factor Price Equalization.

[2] Stolper-Samuelson Theorem.

[3] Rybczinsky Theorem.

[4] Hecksher-Ohlin Theorem.

2 A Small Open Economy

We start by considering a simplified model by focusing on a small open economy. Formally, we analyze a country where goods prices are exogenously given. The assumption can be rationalized by assuming a negligible influence of any country on world prices, entailing that prices behave as if they were parameters for each country.

2.1 Production Functions and CRS

The economy consists of two industries, denoted as good 1 and 2. A key difference relative to the Ricardian model is that the production function combines two production factors, rather than one. We refer to these factors as labor and capital, with the country having a total endowment L^S and K^S of each. We denote the wages and price of capital by w and r.

The production technology of good $i = 1, 2$ is $(l_i, k_i) \mapsto f_i(l, k)$, where $l_i \in \mathbb{R}_+$ and $k_i \in \mathbb{R}_+$ is the labor and capital used by the firm to produce i, respectively. We suppose that f_i is weakly increasing, strictly concave, and exhibits constant returns to scale (henceforth, CRS).

The assumption of CRS puts additional structure to a firm's cost minimization problem. Formally, let $\overline{q}^i$ be the quantity to produce of good i, $l_i(w, k, \overline{q})$ and $k_i(w, k, \overline{q})$ the optimal demand of factors, and $C_i(w, r, \overline{q})$ the minimum cost. Denoting the unit use of factor f of good i by a_{fi}, and the unit cost of good i by $c_i(w, r)$, the assumption of CRS determines that

- $l_i(w, k, \overline{q}) = \overline{q}_i a_{li}(w, r)$, where $a_{li}(w, r) = l_i(w, r, 1)$.
- $k_i(w, k, \overline{q}) = \overline{q}_i a_{ki}(w, r)$, where $a_{ki}(w, r) = k_i(w, r, 1)$
- $C_i(w, r, \overline{q}) = \overline{q}_i c_i(w, r)$, where $c_i(w, r) := C_i(w, r, 1) = w a_{li}(w, r) + r a_{ki}(w, r)$.

Expressed in words, the optimal demand of factors and the minimum cost function are linear in quantities.

Denote the quantity produced in equilibrium of good i by q_i, and its prices by p_i If our goal is to identify the remuneration of factors, (w, r), and the quantities produced in equilibrium, (q_1, q_2), we do not need to characterize the demand side. The rest of the analysis is based on this fact.

Specifically, consider an equilibrium where the country produces all goods (no complete specialization) and all its factors are employed. Then, two set of conditions have to be satisfied in equilibrium. The first one establishes that price has to equal marginal cost:

$$p_1 = wa_{L1}(w,r) + ra_{K1}(w,r), \tag{1a}$$

$$p_2 = wa_{L2}(w,r) + ra_{K2}(w,r), \tag{1b}$$

while the second set reflects the restriction in production imposed by the total endowment of factors:

$$L^S = q_1 a_{L1}(w,r) + q_2 a_{L2}(w,r), \tag{1d}$$

$$K^S = q_1 a_{K1}(w,r) + q_2 a_{K2}(w,r). \tag{1e}$$

Notice that the set of equations (1) is independent of (q_1, q_2), and so they can be used to pin down (w, r). This implies that neither the quantity produced nor the endowment of factors affects the remuneration of factors—a greater factor endowment is only reflected in how much the country produces of each good.

In the next subsections, we analyze how a shock to some parameter affects the equilibrium. The separability of the equilibrium allows us to study (w, r) through (1) exclusively. Furthermore, we can study (q^1, q^2) through (1), once we substitute in the equilibrium values (w, r).

2.2 Factor Price Insensitivity

The system of equations (1) can be expressed in a matrix way by

$$\begin{pmatrix} a_{L1}(w,r) & a_{K1}(w,r) \\ a_{L2}(w,r) & a_{K2}(w,r) \end{pmatrix} \begin{pmatrix} w \\ r \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \tag{4}$$

We denote this system by $\mathbf{A}(\mathbf{w})\,\mathbf{w} = \mathbf{p}$, where $\mathbf{A}(\mathbf{w}) := (a_{fi})_{f\in\{L,K\},i\in\{1,2\}}$, $\mathbf{w} := (w, r)$, and $\mathbf{p} := (p_1, p_2)$. The following result follows by simple inspection of (4).

Proposition: Factor Price Insensitivity

If there is no complete specialization (i.e. both goods are produced in the country), the remuneration of factors are only affected by the prices of goods, but independent of the factors endowments.

2.3 Stolper-Samuelson Theorem

The Stolper-Samuelson Theorem identifies how variations in the prices of goods affect the remuneration of factors. We consider its local version, whereas conditions to make it hold globally are shown in a subsequent section.

Consider an infinitesimal variation in the log-price of each good. This means that $\mathrm{d}\widehat{p}_1 \neq 0$ and $\mathrm{d}\widehat{p}_2 \neq 0$, where $\widehat{\cdot}$ refers to the logarithm natural of a variable. Our goal is to solve for $\mathrm{d}\widehat{w}$ and $\mathrm{d}\widehat{r}$. Differentiating (4),

$$\begin{pmatrix} s_{L1}(w,r) & s_{K1}(w,r) \\ s_{L2}(w,r) & s_{K2}(w,r) \end{pmatrix} \begin{pmatrix} \mathrm{d}\widehat{w} \\ \mathrm{d}\widehat{r} \end{pmatrix} = \begin{pmatrix} \mathrm{d}\widehat{p}_1 \\ \mathrm{d}\widehat{p}_2 \end{pmatrix} \tag{5}$$

where $s_{Li}(w,r) := \frac{w a_{Li}(w,r)}{c_i(w,r)}$ and $s_{Ki}(w,r) := \frac{r a_{Ki}(w,r)}{c_i(w,r)}$ are the cost share of labor and capital in good i. Notice that $s_{Li}(w,r) + s_{Ki}(w,r) = 1$ is always satisfied by definition of costs shares.

Take good i. By the envelope theorem, we know that $\frac{\mathrm{d}c_i(w,r)}{\mathrm{d}w} = a_{Li}(w,r)$. Then, totally differentiating the equation $p_i = c_i(w,r)$, we get

$\mathrm{d}p_i = a_{Li}(w,r)\,\mathrm{d}w + a_{Ki}(w,r)\,\mathrm{d}r$.

This equation can be expressed in logs by multiplying and dividing by the same variable.

$p_i \mathrm{d}\ln p_i = w a_{Li}(w,r)\,\mathrm{d}\ln w + r a_{Ki}(w,r)\,\mathrm{d}\ln r$

Since $p_i = c_i(w,r)$, then

$\mathrm{d}\ln p_i = \frac{w a_{Li}(w,r)}{c_i(w,r)}\mathrm{d}\ln w + \frac{r a_{Ki}(w,r)}{c_i(w,r)}\mathrm{d}\ln r$.

Given $s_{Li}(w,r) := \frac{w a_{Li}(w,r)}{c^i(w,r)}$ and $s_{Ki}(w,r) := \frac{r a_{Ki}(w,r)}{c^i(w,r)}$, and since $s_{Li}(w,r) + s_{Ki}(w,r) = 1$, the equation can be expressed as $\mathrm{d}\ln p_i = s_{Li}(w,r)\,\mathrm{d}\ln w + s_{Ki}(w,r)\,\mathrm{d}\ln r$.

Expressing differentiated system in a matrix way, the result follows.

Let $\mathbf{S}(\mathbf{w}) := (s_{fi})_{f\in\{L,K\},i\in\{1,2\}}$. Thus, (5) can be compactly written as $\mathbf{S}(\mathbf{w})\,\mathbf{d}\widehat{\mathbf{w}} = \mathbf{d}\widehat{\mathbf{p}}$. To get unambiguous results, we need to make an assumption on $\det \mathbf{S}(\mathbf{w})$. Referring to s_{Li}/s_{Ki} as the relative labor cost share in good i, we assume in particular that

$$\det \mathbf{S}(\mathbf{w}) > 0 \Leftrightarrow \frac{s_{L1}(w,r)}{s_{K1}(w,r)} > \frac{s_{L2}(w,r)}{s_{K2}(w,r)},$$

for some given $\mathbf{w}$. This means that the relative labor cost share in good 1 is greater than in good 2.

The assumption can be alternatively stated using technologies as primitives, exploiting that

$$\det \mathbf{A}(\mathbf{w}) > 0 \Leftrightarrow \det \mathbf{S}(\mathbf{w}) > 0,$$

where $\det \mathbf{A}(\mathbf{w}) > 0$ implies that

$$\frac{a_{L1}(w,r)}{a_{K1}(w,r)} > \frac{a_{L2}(w,r)}{a_{K2}(w,r)}.$$

This means that good 1 is relatively intensive in the use of labor at factors prices $\mathbf{w}$. In the case of two goods, notice that good 1 is relatively intensive in the use of labor iff good 2 is intensive in the use of capital. This follows by just reordering the inequality.

By definition, $\det \mathbf{S}(\mathbf{w}) = s_{L1}(w,r)\,s_{K2}(w,r) - s_{K1}(w,r)\,s_{L2}(w,r)$. Besides, using that $s_{Li}(w,r)+s_{Ki}(w,r)=1$ for any i:

$\det S = s_{L1}(w,r)\,[1-s_{L2}(w,r)] - [1-s_{L1}(w,r)]\,s_{L2}(w,r)$

$\Rightarrow \det \mathbf{S}(\mathbf{w}) = s_{L1}(w,r) - s_{L2}(w,r)$.

or, alternatively,

$\det \mathbf{S}(\mathbf{w}) = [1-s_{K1}(w,r)] - [1-s_{K2}(w,r)]$

$\Rightarrow \det \mathbf{S}(\mathbf{w}) = s_{K2}(w,r) - s_{K1}(w,r)$. Therefore,

$$\det \mathbf{S}(\mathbf{w}) = s_{L1}(w,r) - s_{L2}(w,r) = s_{K2}(w,r) - s_{K1}(w,r).$$

Additionally, $\det \mathbf{S}(\mathbf{w}) > 0$ implies that $s_{L1}(w,r) > s_{L2}(w,r)$ and $s_{K2}(w,r) > s_{K1}(w,r)$, ensuring that

$$\frac{s_{L1}(w,r)}{s_{K1}(w,r)} > \frac{s_{L2}(w,r)}{s_{K2}(w,r)}.$$

And using the definition of cost share of labor and capital in good i, which are $s_{Li}(w,r) := \frac{wa_{Li}(w,r)}{c_i(w,r)}$ and

$s_{Ki}(w,r) := \frac{r a_{Ki}(w,r)}{c_i(w,r)}$, then

$$\frac{s_{L1}(w,r)}{s_{K1}(w,r)} > \frac{s_{L2}(w,r)}{s_{K2}(w,r)} \Leftrightarrow \frac{a_{L1}(w,r)}{a_{K1}(w,r)} > \frac{a_{L2}(w,r)}{a_{K2}(w,r)},$$

where the right-hand side is implied by $\det \mathbf{A}(\mathbf{w}) > 0$.

Evaluating (5) in equilibrium, consider $\mathrm{d}\widehat{p}_1 > 0$ and $\mathrm{d}\widehat{p}_2 = 0$, so that the price of the good intensive in labor increases. Then, we obtain the results of the Stolper-Samuelson theorem:

$$\frac{\partial \widehat{w}}{\partial \widehat{p}_1} = \frac{s_{K2}(w,r)}{\det \mathbf{S}(\mathbf{w})} > 1,$$
$$\frac{\partial \widehat{r}}{\partial \widehat{p}_1} = \frac{-s_{L2}(w,r)}{\det \mathbf{S}(\mathbf{w})} < 0.$$

Proposition: Stolper-Samuelson Theorem

If the price of the good intensive in labor increases, then wages increase and the price of capital decreases. Furthermore, the increase in wages is greater than the increase in the price of the good.

Proof. Given, $\mathrm{d}\widehat{p}^1 > 0$ and $\mathrm{d}\widehat{p}^2 = 0$, then $\begin{pmatrix} s_{L1}(w,r) & s_{K1}(w,r) \\ s_{L2}(w,r) & s_{K2}(w,r) \end{pmatrix} \begin{pmatrix} \mathrm{d}\widehat{w} \\ \mathrm{d}\widehat{r} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\widehat{p}^1$.

By Cramer's rule,

$\frac{\partial \widehat{w}}{\partial \widehat{p}^1} = \frac{\det \begin{pmatrix} 1 & s_{K1}(w,r) \\ 0 & s_{K2}(w,r) \end{pmatrix}}{\det \mathbf{S}(\mathbf{w})} \Rightarrow \frac{\partial \widehat{w}}{\partial \widehat{p}^1} = \frac{s_{K2}(w,r)}{\det \mathbf{S}(\mathbf{w})}$

Given $\det \mathbf{S}(\mathbf{w}) = s_{K2}(w,r) - s_{K1}(w,r)$, then $\frac{\partial \widehat{w}}{\partial \widehat{p}_1} = \frac{s_{K2}(w,r)}{s_{K2}(w,r) - s_{K1}(w,r)}$

Since $\det \mathbf{S}(\mathbf{w}) > 0$ by assumption, then $\frac{\partial \widehat{w}}{\partial \widehat{p}_1} = \frac{1}{1 - \frac{s_{K1}(w,r)}{s_{K2}(w,r)}} > 1$

Similar derivation for $\frac{\partial \widehat{r}}{\partial \widehat{p}_1} = \frac{-s_{L2}(w,r)}{\det \mathbf{S}(\mathbf{w})}$. By using that $\det \mathbf{S}(\mathbf{w}) = s_{L1}(w,r) - s_{L2}(w,r)$, $\frac{\partial \widehat{r}}{\partial \widehat{p}_1} = \frac{-s_{L2}(w,r)}{s_{L1}(w,r) - s_{L2}(w,r)} < 0$. ■

The same conclusion can be obtained for a variation in the price of other good. Basically, the theorem states that increases in the price of a good raise the price of the factor intensive in that good, with the opposite happening for the other factor.

The result will eventually have implications for trade liberalization. It will justify that the relatively abundant factor in the exporting industry will have a real income raise.

2.4 Rybczinsky Theorem

The Rybczinsky Theorem characterizes the relation between a variation in the endowment of a factor and the quantity produced of each good. Just like we did with the Stolper-Samuelson theorem, we consider a local version of the result.

Its derivation requires using (1) evaluated at the equilibrium factors prices, $\mathbf{w}^* := (w^*, r^*)$. These prices are identified by (4), and determine that (1) expressed in a matrix way becomes

$$\begin{pmatrix} a_{L1}(\mathbf{w}^*) & a_{L2}(\mathbf{w}^*) \\ a_{K1}(\mathbf{w}^*) & a_{K2}(\mathbf{w}^*) \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} L^S \\ K^S \end{pmatrix}.$$

Considering log-variations, suppose an increase in the endowments of each factor. Then, differentiating the system yields

$$\begin{pmatrix} \lambda_{L1}(\mathbf{w}^*, \mathbf{L}) & \lambda_{L2}(\mathbf{w}^*, \mathbf{L}) \\ \lambda_{K1}(\mathbf{w}^*, \mathbf{L}) & \lambda_{K2}(\mathbf{w}^*, \mathbf{L}) \end{pmatrix} \begin{pmatrix} \mathrm{d}\widehat{q}_1 \\ \mathrm{d}\widehat{q}_2 \end{pmatrix} = \begin{pmatrix} \mathrm{d}\widehat{L}^S \\ \mathrm{d}\widehat{K}^S \end{pmatrix}, \tag{6}$$

where $\lambda_{Li}(\mathbf{w}, \mathbf{L}) := \frac{q_i a^*_{Li}}{L^S}$ and $\lambda_{Ki}(\mathbf{w}, \mathbf{L}) := \frac{q_i a^*_{Ki}}{K^S}$ are evaluated at the equilibrium. These terms represent the share of labor and capital allocated to the production of good i.

We use a superscript $*$ to denote any variable evaluated at $\mathbf{w}^*$. For instance, a^*_{fi} denotes $a_{fi}(\mathbf{w}^*)$.

Consider (1d), since the result for (1e) can be derived analogously. Differentiating (1d),

$\mathrm{d}L^S = a^*_{L1}\,\mathrm{d}q^1 + a^*_{L2}\,\mathrm{d}q^2$.

Multiplying and dividing by each of the variables differentiated,

$\Rightarrow \mathrm{d}\widehat{L}^S L^S = q^1 a^*_{L1}\mathrm{d}\widehat{q}^1 + q^2 a^*_{L2}\mathrm{d}\widehat{q}^2$

$\Rightarrow \mathrm{d}\widehat{L}^S = \frac{q^1 a^*_{L1}}{L^S}\mathrm{d}\widehat{q}^1 + \frac{q^2 a^*_{L2}}{L^S}\mathrm{d}\widehat{q}^2$

Then, using the definition of λ_{fi} for factor f and good i, the result follows.

Our goal is to solve for $\mathrm{d}\widehat{q}_1$ and $\mathrm{d}\widehat{q}_2$. Let $\boldsymbol{\lambda}(\mathbf{w}^*, \mathbf{L}) := (\lambda_{fi})_{f\in\{L,K\}, i\in\{1,2\}}$, so that (6) can be expressed as $\boldsymbol{\lambda}(\mathbf{w}^*, \mathbf{L})\mathbf{d}\widehat{\mathbf{q}} = \mathbf{d}\widehat{\mathbf{L}}$. Just like with Stolper-Samuelson, getting unambiguous comparative statics needs a sign for $\det \boldsymbol{\lambda}(\mathbf{w})$. Referring $\lambda_{Li}/\lambda_{Ki}$ as the relative share of labor allocated to good i, $\det \boldsymbol{\lambda}(\mathbf{w},\mathbf{L}) > 0$ means that the relative share of labor allocated to good 1 is greater than the one allocated to good 2. To show that this indeed holds, we can invoke an assumption that we made previously, since

$$\det \boldsymbol{\lambda}(\mathbf{w},\mathbf{L}) > 0 \Leftrightarrow \det \mathbf{A}(\mathbf{w}) > 0 \Leftrightarrow \det \mathbf{S}(\mathbf{w}) > 0.$$

Therefore, all the results presented so far only require good 1 to be intensive in the use of labor.

The result follows the same steps that we followed to show that $\det \mathbf{A}(\mathbf{w}) > 0 \Leftrightarrow \det \mathbf{S}(\mathbf{w}) > 0$. Specifically,

$$\det \boldsymbol{\lambda}(\mathbf{w},\mathbf{L}) > 0 \Leftrightarrow \frac{\lambda_{L1}(\mathbf{w},\mathbf{L})}{\lambda_{K1}(\mathbf{w},\mathbf{L})} > \frac{\lambda_{L2}(\mathbf{w},\mathbf{L})}{\lambda_{K2}(\mathbf{w},\mathbf{L})},$$

and, since $\lambda_{j1} + \lambda_{j2} = 1$ for $j = L, K$, then

$$\det \boldsymbol{\lambda}(\mathbf{w},\mathbf{L}) = \lambda_{L1}(\mathbf{w},\mathbf{L}) - \lambda_{K1}(\mathbf{w},\mathbf{L}) = \lambda_{K2}(\mathbf{w},\mathbf{L}) - \lambda_{L2}(\mathbf{w},\mathbf{L}).$$

Using the definition of λ_{fi} for $f = L, K$ and $i = 1, 2$, it can be shown that

$$\frac{\lambda_{L1}(\mathbf{w},\mathbf{L})}{\lambda_{K1}(\mathbf{w},\mathbf{L})} > \frac{\lambda_{L2}(\mathbf{w},\mathbf{L})}{\lambda_{K2}(\mathbf{w},\mathbf{L})} \Leftrightarrow \frac{a_{L1}(\mathbf{w})}{a_{K1}(\mathbf{w})} > \frac{a_{L2}(\mathbf{w})}{a_{K2}(\mathbf{w})} \Leftrightarrow \frac{s_{L1}(\mathbf{w})}{s_{K1}(\mathbf{w})} > \frac{s_{L2}(\mathbf{w})}{s_{K2}(\mathbf{w})}.$$

Let us consider in particular a log-increase in the endowment of labor, so that $\mathrm{d}\widehat{L}^S > 0$ and $\mathrm{d}\widehat{K}^S = 0$. Taking (6) and solving for the equilibrium quantities, we arrive to the Rybczinsky Theorem:

$$\frac{\partial \widehat{q}_1}{\partial \widehat{L}^S} = \frac{\lambda_{K2}(\mathbf{w}^*,\mathbf{L})}{\det \boldsymbol{\lambda}(\mathbf{w}^*,\mathbf{L})} > 1,$$
$$\frac{\partial \widehat{q}_2}{\partial \widehat{L}^S} = \frac{-\lambda_{K1}(\mathbf{w}^*,\mathbf{L})}{\det \boldsymbol{\lambda}(\mathbf{w}^*,\mathbf{L})} < 0.$$

Proposition: Rybczinsky Theorem

If the endowment of labor increases, then the quantity of the good intensive in labor increases, while the production of the other good decreases. Furthermore, the increase *in output is greater than the increase in the endowment of the factor.*

Proof.

Given $\mathrm{d}\widehat{L}^S > 0$ and $\mathrm{d}\widehat{K}^S = 0.$, then $\begin{pmatrix} \lambda^*_{L1} & \lambda^*_{L2} \\ \lambda^*_{K1} & \lambda^*_{K2} \end{pmatrix} \begin{pmatrix} \mathrm{d}\widehat{q}_1 \\ \mathrm{d}\widehat{q}_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{d}\widehat{L}^S.$

By Cramer's rule,

$\frac{\partial \widehat{q}_1}{\partial \widehat{L}^S} = \frac{\det \begin{pmatrix} 1 & \lambda^*_{L2} \\ 0 & \lambda^*_{K2} \end{pmatrix}}{\det \lambda^*} \Rightarrow \frac{\partial \widehat{q}_1}{\partial \widehat{L}^S} = \frac{\lambda^*_{K2}}{\det \boldsymbol{\lambda}^*}$

Given $\det \boldsymbol{\lambda}^* = \lambda^*_{K2} - \lambda^*_{L2}$ then $\frac{\partial \widehat{q}_1}{\partial \widehat{L}^S} = \frac{\lambda^*_{K2}}{\lambda^*_{K2} - \lambda^*_{L2}}$.

Since $\det \boldsymbol{\lambda}^* > 0$ by assumption, then $\frac{\partial \widehat{q}_1}{\partial \widehat{L}^S} = \frac{1}{1 - \frac{\lambda^*_{L2}}{\lambda^*_{K2}}} > 1.$

Similar derivation for $\frac{\partial \widehat{q}_2}{\partial \widehat{L}^S} = \frac{-\lambda^*_{K1}}{\det \boldsymbol{\lambda}^*} < 0.$ ■

2.5 Global Validity of the Theorems

The Stolper-Samuelson and Rybczinsky theorem were derived assuming small shocks. Both theorems hold if $\det \mathbf{A}(\mathbf{w}) > 0$ at the equilibrium value $\mathbf{w}^*$. Locally, the assumption is quite mild. It only demands that one of the goods is intensive in the use of a factor—we can always relabel the goods and still get the same sign for the determinant if $\det \mathbf{A}(\mathbf{w}) < 0$.

By assuming $\det \mathbf{A}(\mathbf{w})$ for any $\mathbf{w}$, the conclusion can be easily extend to hold for arbitrary changes in the parameters. However, extending the assumption to hold globally is way stronger, requiring that technologies do not exhibit factor-intensity reversals. This means that a good is always intensive in the use of the same factor, irrespective of whether the price of that factor becomes disproportionately high or low. Basically, we are restricting the impact of factor prices in the substitution of factors.

3 Trade Between Two Big Economies

With the results for a small open economy, we can now think about trade between two big economies. Just like Ricardo and any other Neoclassical model, Heckscher-

Ohlin ensures positive gains of trade in each country. On the contrary, in contrast to Ricardo, the existence of more than one factor allows us to think about how these gains are redistributed. The main conclusion will be that trade liberalization always creates winners and losers.

We consider two countries, (H) and (F), which we label "home" and "foreign". Following the notation used for the Ricardian model, any variable without a tilde refers to (H), while a variable with a tilde refers to (F). The model is based on the following assumptions

- Factors endowments are given. They are not affected by or investments (including education, which would improve the workers' skills) or by changes in market conditions (e.g. additional workers do not search for jobs if there are better job opportunities).
- Countries have access to the same technologies.
- Factors are mobile between industries in the country, but immobile across countries.
- All goods are tradable, with no trade costs or other type of trade friction.
- Preferences are homothetic and identical in each country.

The assumption of identical technologies rules out an explanation of trade resembling Ricardo. This enables us to starkly show the role of differences in endowments.

When preferences are homothetic, the consumption of good i can be expressed as $q_i\left(y, \frac{p_i}{p_j}\right) = y h_i\left(\frac{p_i}{p_j}\right)$ for $j \neq i$, where y is the agent's income and h_i is a function that satisfies $h_i' < 0$ and $h_j' > 0$. The result implies that

$$\frac{q_i\left(y, \frac{p_i}{p_j}\right)}{q_j\left(y, \frac{p_i}{p_j}\right)} = \frac{h_i\left(\frac{p_i}{p_j}\right)}{h_j\left(\frac{p_i}{p_j}\right)},$$

so that the relative demand is independent of income and decreases when $\frac{p_i}{p_j}$ is higher.

Finally, we add some assumptions about the intensity of factors and relative endowments. They hold without loss of generality, as we can always relabel countries and goods to obtain the same implications.

[1] (H) is relatively abundant in labor, which given two countries implies that (F) is relatively abundant in capital. Formally,

$$\frac{L}{K} > \frac{\widetilde{L}}{\widetilde{K}}$$

[2] Good 1 is relatively intensive in the use of labor, which given two goods implies that good 2 is relatively intensive in the use of capital. Furthermore, there are no factor-intensity reversals; formally, $\det \mathbf{A}(\mathbf{w}) > 0$ for any $\mathbf{w}$, so that

$$\frac{a_{L1}(\mathbf{w})}{a_{K1}(\mathbf{w})} > \frac{a_{L2}(\mathbf{w})}{a_{K2}(\mathbf{w})}.$$

Notice we are focusing on global results, since we are assuming no factor-intensity reversals.

3.1 Equilibrium

Consider the free-trade case. Our first result is derived by simple observation, although its implications are strong. It is based on that the law of one price holds for each good under trade, in the absence of trade costs. Observe that, keeping aside that prices are now endogenously determined because there are two big countries, equilibrium in each country still requires that (1) and (1) hold. Therefore, taking into account (1) and due to the law of one price, the following result holds.

Proposition: Factor Price Equalization

Suppose that both countries diversify their production (i.e. no complete specialization). Then, the remuneration of each factor is exclusively determined by the prices of the goods, and thus independent of the factors endowments. As a corollary, the law of one price entails that both countries have the same remuneration of factors under trade.

The following lemma also characterizes free trade, and is a consequence of the Stolper-Samuelson theorem.

Lemma 3.1 Under free trade and relative to its trading partner, each country produces more of the good intensive in its relatively abundant factor.

Proof. Under free trade, the law of one price holds for each good, and so the remuneration of factors is the same in each country, given factor-price equalization. Based on this, the strategy of the proof consists of two steps. Take one of the countries, and define $l := \frac{L^S}{K^S}$ and $q := \frac{q_1}{q_2}$. The first step is to show that the optimal relative quantities can be expressed as $q^*(l)$. The second step is to show that q^* is monotonically increasing. This proves the result, since $l > \widetilde{l}$ and hence $q(l) > q\left(\widetilde{l}\right)$.

As for the first step, the equilibrium has to satisfy (1) for each country. Take one of the countries, and divide (1d) by (1e), so that $l = \frac{q_1 a_{L1}(w,r)+q_2 a_{L2}(w,r)}{q_1 a_{K1}(w,r)+q_2 a_{K2}(w,r)}$ and hence

$$\ln l = \ln\left(q a_{L1}(w,r) + a_{L2}(w,r)\right) - \ln\left(q a_{K1}(w,r) + a_{K2}(w,r)\right). \tag{7}$$

Let $q^*(l)$ the value of q that solves (7), which concludes the first step.

Differentiating (7), we obtain $\mathrm{d}\ln l = \left[\frac{a^*_{L1}}{q a^*_{L1}+a^*_{L2}} - \frac{a^*_{K1}}{q a^*_{K1}+a^*_{K2}}\right]\mathrm{d}q^*$, or equivalently

$$\mathrm{d}\ln l = \left[\frac{a^*_{L1}a^*_{K2} - a^*_{K1}a^*_{L2}}{\left(q^* a^*_{L1} + a^*_{L2}\right)\left(q^* a^*_{K1} + a^*_{K2}\right)}\right]\mathrm{d}q^*,$$

Letting $\kappa(l, \mathbf{w}) := \left[\frac{a^*_{L1}a^*_{K2} - a^*_{K1}a^*_{L2}}{\left(q^* a^*_{L1} + a^*_{L2}\right)\left(q^* a^*_{K1} + a^*_{K2}\right)}\right]$, notice that $\kappa(l, \mathbf{w}) > 0$ since $\det \mathbf{A}(\mathbf{w}) > 0$. Therefore, $\frac{\mathrm{d}q^*}{\mathrm{d}l} = \frac{l}{\kappa} > 0$, which concludes the second step.■

The lemma implies that, under free trade, the country with a greater relative endowment of labor produces relatively more of good 1 (the labor-intensive good). On the contrary, the other country produces relatively more of good 2 (the capital-intensive good), as it is relatively more endowed in capital.

We now proceed to characterize trade between countries. This leads us to the Hecksher-Ohlin theorem, whose proof is based on the lemma just derived.

Proposition: Heckscher-Ohlin Theorem

Each country exports the good that is intensive in its relatively abundant factor.

Proof. To state the result, we assume that trade is balanced, and add the equilibrium condition for each good. We keep denoting supply of good i in (H) and $(F)$by q_i and $\widetilde{q}_i$, respectively, while demand is denoted x_i and $\widetilde{x}_i$. Equilibrium requires that supply equals demand for each good.

The proof is by contradiction. Recall that country (H) is relatively abundant in labor, and good 1 is intensive in the use of labor. So, towards a contradiction, suppose, $x_1^* \geq q_1^*$ and $x_2^* \leq q_2^*$. Thus, (H) imports good 1 and exports good 2. This implies that $\widetilde{x}_1^* \leq \widetilde{q}_1^*$ and $\widetilde{x}_2^* \geq \widetilde{q}_2^*$ for country (F), implying that it imports good 2 and exports good 1. By making use of the inequalities obtained, then $\frac{x_1^*}{x_2^*} \geq \frac{q_1^*}{q_2^*}$ and $\frac{\widetilde{q}_1^*}{\widetilde{q}_2^*} \geq \frac{\widetilde{x}_1^*}{\widetilde{x}_2^*}$.

By the lemma we proved before, $\frac{q_1^*}{q_2^*} > \frac{\widetilde{q}_1^*}{\widetilde{q}_2^*}$, and homotheticity of preferences implies that $\frac{x_1^*}{x_2^*} = \frac{\widetilde{x}_1^*}{\widetilde{x}_2^*}$. Therefore,

$$\frac{x_1^*}{x_2^*} \geq \frac{q_1^*}{q_2^*} > \frac{\widetilde{q}_1^*}{\widetilde{q}_2^*} \geq \frac{\widetilde{x}_1^*}{\widetilde{x}_2^*}$$

which is a contradiction given $\frac{x_1^*}{x_2^*} = \frac{\widetilde{x}_1^*}{\widetilde{x}_2^*}$.■

3.2 Gains of Trade and Distribution of Income

We conclude the analysis by studying gains of trade and their distribution across agents. Regarding the former, Heckscher-Ohlin is a Neoclassical model. In a previous lecture note, we proved that these models always entail positive gains of trade. This means that trade liberalization expands each country's consumption possibilities, relative to autarky.

Unlike Ricardo, which assumes only one production factor, we can go further and analyze how these gains for the country translate into each agent's welfare. Specifically, Heckscher-Ohlin assumes labor and capital as production factors. These factors contribute differently to the production process, and hence earn different remunerations. This implies that agents do not equally benefit from trade. In fact, the main conclusion of the model is that trade always creates winners and losers.

To show this formally, we need to characterize the autarky equilibrium in each country. For country (H) and (F), denote the relative price of good 1 in autarky by $p^a := \frac{p_1^a}{p_2^a}$ and $\widetilde{p}^a := \frac{\widetilde{p}_1^a}{\widetilde{p}_2^a}$, respectively. Moreover, let the relative price of good 1 under free trade be $p^* := \frac{p_1^*}{p_2^*}$. The following two lemmas characterize the autarky relative prices and how they compare to the relative prices under trade.

Lemma 3.2 $\frac{p_1^a}{p_2^a} < \frac{\widetilde{p}_1^a}{\widetilde{p}_2^a}$

Proof. Towards a contradiction, suppose $\widetilde{p}^a = p^a$. By the Factor Price Equalization, this implies that the remuneration of factors under autarky is the same in each country. Given identical homothetic preferences in each country, $x^a := \frac{x_1^a}{x_2^a} = \frac{\widetilde{x}_1(p^a)}{\widetilde{x}_2(p^a)} =: \widetilde{x}(p^a)$. Moreover, given equal remuneration of factors, $q^a > \widetilde{q}^a$ by the Rybczinsky Theorem. Since $\widetilde{x}(p^a) = x^a = q^a > \widetilde{q}^a$, then $\widetilde{x}(p^a) > \widetilde{q}^a$. This is the same as $\frac{\widetilde{x}_1(p^a)}{\widetilde{x}_2(p^a)} > \frac{\widetilde{q}_1^a}{\widetilde{q}_2^a}$, which implies that $\frac{\widetilde{x}_1(p^a)}{\widetilde{q}_1^a} > \frac{\widetilde{x}_2(p^a)}{\widetilde{q}_2^a}$. Therefore, $\widetilde{p}^a \neq p^a$ and there has to be an excess of demand in (F) for either good 1 or good 2.
We now simplify notation by skipping the arguments of the functions. Towards a contradiction, suppose (F) does not have an excess of demand for good 1 at relative prices p^a, so that $\widetilde{q}_1 \geq \widetilde{x}_1$. We want to show that this leads us to a contradiction, and so $\widetilde{q}_1 < \widetilde{x}_1$. Since $\frac{\widetilde{x}_1}{\widetilde{q}_1^a} > \frac{\widetilde{x}_2}{\widetilde{q}_2^a}$ and $\frac{\widetilde{x}_1}{\widetilde{q}_1^a} \leq 1$, it is true that $\frac{\widetilde{x}_2}{\widetilde{q}_2^a} < 1$. But then, $\frac{\widetilde{x}_2}{\widetilde{q}_2^a} < 1$ and $\frac{\widetilde{x}_1}{\widetilde{q}_1^a} \leq 1$, so that $\widetilde{x}_2 < \widetilde{q}_2^a$ and $\widetilde{x}_1 \leq \widetilde{q}_1^a$. Hence, $\frac{\widetilde{q}_1^a}{\widetilde{x}_2} > \frac{\widetilde{x}_1}{\widetilde{q}_2^a}$ by using these inequalities, which implies $\frac{\widetilde{q}_1^a}{\widetilde{x}_1} > \frac{\widetilde{x}_2}{\widetilde{q}_2^a}$ and so $\frac{\widetilde{x}_1}{\widetilde{q}_1^a} < \frac{\widetilde{x}_2}{\widetilde{q}_2^a}$, which is a contradiction.
Thus, we have proved that (F) has an excess of demand for good 1 at relative prices p^a. Therefore, the relative price has to be higher to restore the equilibrium.[a] Therefore, $p^a < \widetilde{p}^a$. ■

[a]With homothetic preferences, the uncompensated law of demand holds, and so an increase in p^a always reduces the relative demand of good 1.

Lemma 3.3 $\frac{p_1^a}{p_2^a} < \frac{p_1^*}{p_2^*} < \frac{\widetilde{p}_1^a}{\widetilde{p}_2^a}$

Proof. Let $z(p) := c_1(p) - q_1(p)$ and $\widetilde{z}(p) := \widetilde{c}_1(p) - \widetilde{q}_1(p)$ be the excess demand of good 1 in country (H) and (F) when relative prices are p. By definition of autarky prices, $z(p^a) = 0$, and we showed in the previous lemma that $\widetilde{z}(p^a) > 0$. Thus, $z(p^a) + \widetilde{z}(p^a) > 0$ and there is a world excess of demand at $p^* = p^a$. Similarly, $z(\widetilde{p}^a) < 0$ and $\widetilde{z}(\widetilde{p}^a) = 0$, so that $z(p^a) + \widetilde{z}(p^a) < 0$ and there is a world excess of supply at $p^* = \widetilde{p}^a$.
Hence, by the intermediate value theorem and the continuity of the excess demand (in addition to the prices set defined in a convex set) $\exists p^*$ such that $z(p^*) + \widetilde{z}(p^*) = 0$ and $p^a < p^* < \widetilde{p}^a$.■

The lemmas inform us what occurs under trade relative to autarky: the good with more export opportunities experiences a greater increase in relative price, while the good subject to tougher import competition experiences a decrease in relative price.

In the model, we measure each agent's welfare in terms of real income. and hence the level of consumption. To determine this, we need to compare these changes in prices with the variations in each agent's income. And since nominal income and prices change simultaneously, we need to determine whether income has increased or decreased relative to each good's price.

The proof requires two steps. In the first step, we start considering an infinitesimal variation in log-prices starting from autarky, i.e. $\mathrm{d}\widehat{p}_1 \neq 0$ and $\mathrm{d}\widehat{p}_2 \neq 0$. The goal is to show that the relatively abundant factor always has an increase in income, and this is higher than the variation in prices. For instance, consider the home country, which is relatively abundant in labor. Let the variation in remunerations be $\mathrm{d}\widehat{w}$ and $\mathrm{d}\widehat{r}$. We know that labor gains from trade and capital loses for sure if $\mathrm{d}\widehat{w} > \sup\{\mathrm{d}\widehat{p}_1, \mathrm{d}\widehat{p}_2\}$ and $\inf\{\mathrm{d}\widehat{p}_1, \mathrm{d}\widehat{p}_2\} > \mathrm{d}\widehat{r}$. To see why this is so, notice that $\frac{p_1^a}{p_2^a} < \frac{p_1^*}{p_2^*}$ by the lemmas proven, which implies that $\mathrm{d}\widehat{p}_1 > \mathrm{d}\widehat{p}_2$. Thus, $\mathrm{d}\widehat{w} > \mathrm{d}\widehat{p}_1 > \mathrm{d}\widehat{p}_2$ and $\mathrm{d}\widehat{p}_1 > \mathrm{d}\widehat{p}_2 > \mathrm{d}\widehat{r}$. This is what Jones (1965) calls "the magnification effect", where $\mathrm{d}\widehat{w} > \mathrm{d}\widehat{p}_1 > \mathrm{d}\widehat{p}_2 > \mathrm{d}\widehat{r}$. The second step requires using this result in a global way.

Regarding the first step, differentiating (1),

$$\begin{pmatrix} s_{L1}(w,r) & s_{K1}(w,r) \\ s_{L2}(w,r) & s_{K2}(w,r) \end{pmatrix} \begin{pmatrix} \mathrm{d}\widehat{w} \\ \mathrm{d}\widehat{r} \end{pmatrix} = \begin{pmatrix} \mathrm{d}\widehat{p}_1 \\ \mathrm{d}\widehat{p}_2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \mathrm{d}\widehat{w} \\ \mathrm{d}\widehat{r} \end{pmatrix} = \frac{1}{s_{L1}(\mathbf{w}) - s_{L2}(\mathbf{w})} \begin{pmatrix} s_{K2}(\mathbf{w}) & -s_{K1}(\mathbf{w}) \\ -s_{L2}(\mathbf{w}) & s_{L1}(\mathbf{w}) \end{pmatrix} \begin{pmatrix} \mathrm{d}\widehat{p}_1 \\ \mathrm{d}\widehat{p}_2 \end{pmatrix}.$$

This determines the following solutions for country (H)

$$\mathrm{d}\widehat{w} = \frac{s_{K2}(\mathbf{w})\,\mathrm{d}\widehat{p}_1 - s_{K1}(\mathbf{w})\,\mathrm{d}\widehat{p}_2}{s_{K2}(\mathbf{w}) - s_{K1}(\mathbf{w})}, \tag{8a}$$

$$\mathrm{d}\widehat{r} = \frac{-s_{L2}(\mathbf{w})\,\mathrm{d}\widehat{p}_1 + s_{L1}(\mathbf{w})\,\mathrm{d}\widehat{p}_2}{s_{L1}(\mathbf{w}) - s_{L2}(\mathbf{w})}, \tag{8b}$$

where we have used that $s_{L1}(\mathbf{w}) - s_{L2}(\mathbf{w}) = s_{K2}(\mathbf{w}) - s_{K1}(\mathbf{w})$. With this result, we can proceed to the second step, which shows existence of winners and losers following trade liberalization.

Proposition: Winners and Losers under Trade

Relative to autarky and measuring welfare through real income, the relatively abundant factor gains while the non-abundant loses following trade liberalization.

Proof.
We prove the result for the home country. Relabeling countries and goods, the proof also applies to the foreign country.
We first need to show that $\mathrm{d}\widehat{p}_1 > \mathrm{d}\widehat{p}_2$. We use that
$\mathrm{d}\widehat{p}_1 > \mathrm{d}\widehat{p}_2 \Leftrightarrow \mathrm{d}\left(\widehat{p}_1 - \widehat{p}_2\right) > 0 \Leftrightarrow \mathrm{d}\left[\ln\left(\frac{p_1^*}{p_1^a}\right) - \ln\left(\frac{p_2^*}{p_2^a}\right)\right] > 0$
$\Leftrightarrow \mathrm{d}\ln\left(\frac{p_1^*/p_1^a}{p_2^*/p_2^a}\right) > 0 \Leftrightarrow \mathrm{d}\ln\left(\frac{p_1^*/p_2^*}{p_1^a/p_2^a}\right) > 0 \Leftrightarrow \mathrm{d}\ln\left(\frac{p^*}{p^a}\right) > 0.$
Since we already proved in the lemmas that $\frac{p_1^a}{p_2^a} < \frac{p_1^*}{p_2^*}$, the result follows.
Now, we will show that $\mathrm{d}\widehat{w} > \mathrm{d}\widehat{p}_1$ and $\mathrm{d}\widehat{p}_2 > \mathrm{d}\widehat{r}$. If this holds, then the result follows, since it implies that $\mathrm{d}\widehat{w} > \mathrm{d}\widehat{p}_1 > \mathrm{d}\widehat{p}_2$ and $\mathrm{d}\widehat{p}_1 > \mathrm{d}\widehat{p}_2 > \mathrm{d}\widehat{r}$.
Equation (8a) can be reexpressed as
$\mathrm{d}\widehat{w} - \mathrm{d}\widehat{p}_1 = \frac{s_{K2}(\mathbf{w})\mathrm{d}\widehat{p}_1 - s_{K1}(\mathbf{w})\mathrm{d}\widehat{p}_2}{s_{K2}(\mathbf{w}) - s_{K1}(\mathbf{w})} - \mathrm{d}\widehat{p}_1$
$\Rightarrow \mathrm{d}\widehat{w} - \mathrm{d}\widehat{p}_1 = \frac{s_{K2}(\mathbf{w})\mathrm{d}\widehat{p}_1 - s_{K1}(\mathbf{w})\mathrm{d}\widehat{p}_2 - s_{K2}(\mathbf{w})\mathrm{d}\widehat{p}_1 + s_{K1}(\mathbf{w})\mathrm{d}\widehat{p}_1}{s_{K2}(\mathbf{w}) - s_{K1}(\mathbf{w})}$
$\Rightarrow \mathrm{d}\widehat{w} - \mathrm{d}\widehat{p}_1 = \frac{s_{K1}(\mathbf{w})(\mathrm{d}\widehat{p}_1 - \mathrm{d}\widehat{p}_2)}{s_{K2}(\mathbf{w}) - s_{K1}(\mathbf{w})}$
Since by assumption $\mathrm{d}\widehat{p}_1 - \mathrm{d}\widehat{p}_2 > 0$ and $s_{K2}(\mathbf{w}) - s_{K1}(\mathbf{w}) > 0$, then $\mathrm{d}\widehat{w} > \mathrm{d}\widehat{p}_1$.
Moreover,
$\mathrm{d}\widehat{r} - \mathrm{d}\widehat{p}_2 = \frac{-s_{L2}(\mathbf{w})\mathrm{d}\widehat{p}_1 + s_{L1}(\mathbf{w})\mathrm{d}\widehat{p}_2}{s_{L1}(\mathbf{w}) - s_{L2}(\mathbf{w})} - \mathrm{d}\widehat{p}_2$
$\Rightarrow \mathrm{d}\widehat{r} - \mathrm{d}\widehat{p}_2 = \frac{-s_{L2}(\mathbf{w})\mathrm{d}\widehat{p}_1 + s_{L1}(\mathbf{w})\mathrm{d}\widehat{p}_2 - s_{L1}(\mathbf{w})\mathrm{d}\widehat{p}_2 + s_{L2}(\mathbf{w})\mathrm{d}\widehat{p}_2}{s_{L1}(\mathbf{w}) - s_{L2}(\mathbf{w})}$
$\Rightarrow \mathrm{d}\widehat{r} - \mathrm{d}\widehat{p}_2 = \frac{-s_{L2}(\mathbf{w})[\mathrm{d}\widehat{p}_1 - \mathrm{d}\widehat{p}_2]}{s_{L1}(\mathbf{w}) - s_{L2}(\mathbf{w})}$,
which then implies that $\mathrm{d}\widehat{r} < \mathrm{d}\widehat{p}_2$.■

The result can also be interpreted through the lens of the changes in each industry, following trade liberalization. It means that the intensive factor in the good that has more export opportunities gains from trade, while the intensive factor in the good subject to tougher import competition loses from trade.