International Trade[1]

Lecture Note 3: Ricardo with a Continuum of Goods and n Countries

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[1]The notes are still preliminary and in beta. Please, if you find any typo or mistake, send it to malfaro@ualberta.ca.

Contents

Notation

This is a derivation

This is some comment

This is a comment on advanced topics which are not part of the course (you can ignore it without loss of continuity regarding the text)

- The symbol ":=" means "by definition".
- I denote vectors by bold lowercase letters (for instance, $\mathbf{x}$) and matrices by bold capital letters (for instance, $\mathbf{X}$).
- To differentiate between the verb "maximize" and the operator "maximum", I denote the former with "max" and the latter with "sup" (i.e., supremum). The same caveat applies to "minimize" and "minimum", where I use "min" and "inf", with the latter indicating infimum.
- "iff" means "if and only if"
- $\exp(x)$ is the function e^x.
- Random variables are denoted with a bar below. For instance, $\underline{x}$.

These notes contain hyperlinks in blue and red text. If you are using Adobe Acrobat Reader, you can click on the link and easily navigate back by pressing Alt+Left Arrow.

1 Problems of Generalization

Ricardo's original model considers two countries and two goods. In the previous lecture note, we relaxed this assumption by extending the model to handle a continuum of goods. However, we still supposed the existence of two countries.

The model with a continuum of goods yields similar conclusions for the patterns of production, where countries produce and export goods based on comparative advantages. Although we did not prove it, it can be shown that the same conclusions arise if we extend the model to more than two countries, as long as there are two goods.

The reason why similar conclusions hold in all these variants is that they allow for a clear ranking of each country's efficiency. Consequently, the mechanisms of each model are ultimately the same. On the contrary, generalizing the Ricardian model to simultaneously allow for multiple goods and countries is not straightforward. In fact, it invalidates several of the conclusions. For instance, (bilateral) comparative advantages are not sufficient to identify patterns of production and trade anymore.

To illustrate this issue, consider the following example from Jones (1961). He envisions a world with three countries and three goods, with the following unit labor requirements:

Figure 1: *Unit Labor Requirements*

Country	(A)	(B)	(C)
Good 1	10	10	10
Good 2	5	7	3
Good 3	4	3	2

Suppose the following pattern of specialization:

- (C) in good 3,
- (A) in good 2, and
- (B) in good 1.

Given these specializations, each country has a bilateral comparative advantage in the good it produces.[1] Nonetheless, it can be shown that this pattern of production cannot be part of a competitive solution under free trade.[2]

1.1 Jones (1961) (OPTIONAL)

Jones (1961) provides an approach to identifying the equilibrium when there is complete specialization in each country. He defines an ***assignment*** as a pattern of complete specialization, where labor in each country is completely allocated to the production of one good. He additionally defines a ***class of assignments***, as the collection of assignments with the same number of countries specialized in each good. Given these defintiions, Jones (1961) identifies the optimal assignment in its class. Then, he pins down the equilibrium by using that a competitive solution needs to be optimal, as shown by McKenzie (1954).

The conclusions he obtains are the following. First, excluding cases where two countries have exactly the same relative costs, there is a unique optimal assignment in each class. Optimality is defined relative to the world efficiency locus. Additionally, the optimal solution needs to minimize the product of labor coefficients involved in the assignment. In a two-by-two world, this coincides with comparative advantages. More generally, though, while bilateral comparative advantages need to hold in the optimal allocation, this is not sufficient to get an equilibrium.

To see this, consider again the example in Figure 1. The pattern of specialization with bilateral comparative advantages considered above is indeed inefficient, even though it is optimal. And, by using the approach of Jones (1961), it can be shown that the optimal assignment of the class is (A) to good 1, (B) to good 3, and (C) to good 2. In that case, the product of unit labor requirements is $10 \times 3 \times 3 = 90$, while in the other assignment would be $10 \times 5 \times 2 = 100$.

[1] Given a pattern of production which is a candidate to an equilibrium, bilateral comparative advantages are defined according to the good that each country is producing. In other terms, bilateral comparative advantages do not require a comparison of relative efficiencies for goods different to the ones assigned to each country. In the example considered, let $a_i(j)$ be the unit labor requirement of country i for good j. Then, (C) relative to (A) satisfies $\frac{a_C(3)}{a_A(3)} = \frac{2}{4} < \frac{a_C(2)}{a_A(2)} = \frac{3}{5}$ and relative to (B), $\frac{a_C(3)}{a_B(3)} = \frac{2}{3} < \frac{a_C(1)}{a_A(1)} = \frac{10}{10}$, while (B) relative to (C), $\frac{a_B(1)}{a_C(1)} = \frac{10}{10} < \frac{a_B(3)}{a_C(3)} = \frac{3}{2}$. Thus each country has a bilateral comparative advantage in the good that it is producing.

[2] In particular, McKenzie (1954) shows that this pattern of production is not world efficient and, so, it can be shown that it cannot consittute a competitive solution.

The set of optimal assignments determines the world efficiency frontier. And, given some demand conditions, one of the points along this frontier constitutes the actual production in a competitive solution.

2 Intuitions Behind EK

Eaton and Kortum (2002) (henceforth, EK) is an extension of DFS model to an arbitrary number of countries. This paper has had a profound impact on the trade literature of the last decades. It generalizes the DFS model, and also provides a methodology to take the Ricardian model to the data.

EK generalize Ricardo by employing a probabilistic approach for the concept of comparative advantages. Preserving the importance of comparative advantages is not without any cost: the model is now silent about which specific goods will be produced and traded by each country. Instead, the model only identifies the fraction of goods that a country produces, along with the total bilateral trade flows between countries. Nonetheless, even when we cannot obtain sharp predictions for these matters, the model can be used to estimate gains of trade in particular.

2.1 Relation Between DFS (1977) and EK (2002)

To provide some intuition for EK, let's begin by establishing a link with the DFS model. We do this by showing that DFS is a special case of EK, consisting of two countries and a specific functional form for the relative efficiencies A.

The key issue in extending the Ricardian model is the breakdown of a natural order for goods, which serves as the basis for comparative advantages. EK makes use of a probabilistic approach to overcome this challenge, rendering the relative-efficiency order irrelevant for deriving results.

As in DFS, let's consider two countries (H) and $[F]$, where a tilde is used to indicate foreign variables. Furthermore, we consider a continuum of goods, whose set is given by $J := [0, 1]$.

DFS describes the technology of production by the unit labor requirements. Instead, EK exploits that marginal productivity conveys the same information when there is one production factor. Denoting the unit labor requirements by a and the marginal productivity labor by z, the relation between both concepts is $z = \frac{1}{a}$.

EK interprets each country's labor productivity, z and $\widetilde{z}$, as realizations of random variables $\underline{z}$ and $\underline{\widetilde{z}}$, respectively. This feature prevents us from making deterministic predictions about a specific draw. In other words, the model is silent about a country's efficiency for a particular good. However, the model is still capable of describing what we could expect for a country's overall production. This is possible due to the Law of Large Numbers and the assumption of an infinite number of goods, which ensures that what we expect is what actually occurs.

As EK's ultimate goal is to use the model empirically, they assume specific distributions for the random marginal productivities. Specifically, they suppose that each $\underline{z}(j)$ and $\underline{\widetilde{z}}(j)$ is drawn from a Fréchet distribution, and so the cdfs are respectively

$$\begin{aligned} F(z) &:= \exp\left[-Tz^{-\theta}\right], \\ \widetilde{F}(\widetilde{z}) &:= \exp\left[-\widetilde{T}\widetilde{z}^{-\theta}\right], \end{aligned}$$

where the parameters T and $\widetilde{T}$ are country specific, but θ is common across countries. In turn, the corresponding pdfs are

$$\begin{aligned} f(z) &:= \exp\left[-Tz^{-\theta}\right]\theta T z^{-\theta-1}, \\ f(z) &:= \exp\left[-\widetilde{T}\widetilde{z}^{-\theta}\right]\theta\widetilde{T}\widetilde{z}^{-\theta-1}. \end{aligned}$$

Focusing on country (H), a higher value of T indicates that (H) has higher efficiency draws on average. Thus, **the parameter T reflects absolute advantages**. Likewise, a higher θ means that draws are less dispersed**, turning θ a parameter capturing comparative advantages.** The intuition behind is that a high dispersion of productivity draws implies that the draws are more dissimilar on average. As a result, we should expect larger differences in productivity across countries.

By restating the model in probabilistic terms, we can reinterpret the function $A(j)$. Given goods $J := [0, 1]$, j can be understood as the fraction of goods for which the expected relative efficiency is greater than some value A. Formally, $j = \Pr\left[\frac{\underline{z}}{\underline{\widetilde{z}}} > A\right]$, and so

$$j = \frac{TA^{-\theta}}{TA^{-\theta} + \widetilde{T}}.$$

This makes it possible to derive $A(j)$, which is the relative efficiency of some given fraction of goods:

$$A(j) = \left(\frac{T}{\widetilde{T}}\right)^{\frac{1}{\theta}} \left(\frac{1-j}{j}\right)^{\frac{1}{\theta}}.$$

Suppose some value A. We want to know the fraction of goods such that the relative productivity of countries is A or more. Formally, this means, $j = \Pr\left[A < \frac{\widetilde{a}}{a}\right]$ and expressing it in terms of marginal productivities,

$j = \Pr\left[A < \frac{\widetilde{a}}{a}\right] \Rightarrow j = \Pr\left[A\widetilde{z} < z\right]$

$\Rightarrow j = 1 - \Pr\left[z < A\widetilde{z}\right] \Rightarrow j = 1 - \int_{\widetilde{z}} \exp\left[-T\widetilde{z}^{-\theta}A^{-\theta}\right] \mathrm{d}\widetilde{F}$

And note that $\widetilde{f}(\widetilde{z}) = \exp\left[-\widetilde{T}\widetilde{z}^{-\theta}\right] \theta\widetilde{T}\widetilde{z}^{-\theta-1}$.

So $j = 1 - \int_{\widetilde{z}} \exp\left[-T\widetilde{z}^{-\theta}A^{-\theta}\right] \exp\left[-\widetilde{T}\widetilde{z}^{-\theta}\right] \theta\widetilde{T}\widetilde{z}^{-\theta-1}\mathrm{d}\widetilde{z}$

$\Rightarrow 1 - j = \widetilde{T} \int_{\widetilde{z}} \exp\left[-T\widetilde{z}^{-\theta}A^{-\theta} - \widetilde{T}\widetilde{z}^{-\theta}\right] \theta\widetilde{z}^{-\theta-1}\mathrm{d}\widetilde{z}$

and noting that $\frac{TA^{-\theta}+\widetilde{T}}{TA^{-\theta}+\widetilde{T}} = 1$

$\Rightarrow 1 - j = \widetilde{T} \int_{\widetilde{z}} \exp\left[-\left(TA^{-\theta}+\widetilde{T}\right)\widetilde{z}^{-\theta}\right] \theta\widetilde{z}^{-\theta-1}\frac{TA^{-\theta}+\widetilde{T}}{TA^{-\theta}+\widetilde{T}}\mathrm{d}\widetilde{z}$

$\Rightarrow 1 - j = \frac{\widetilde{T}}{TA^{-\theta}+\widetilde{T}} \int_{\widetilde{z}} \exp\left[-\left(TA^{-\theta}+\widetilde{T}\right)\widetilde{z}^{-\theta}\right] \theta\left(TA^{-\theta}+\widetilde{T}\right)\widetilde{z}^{-\theta-1}\mathrm{d}\widetilde{z}$

Since the integral is equal to 1 (the function to be integrated is the density of a Fréchet with constant $\left(TA^{-\theta}+\widetilde{T}\right)$) then

$1 - j^* = \frac{\widetilde{T}}{TA^{-\theta}+\widetilde{T}}$ and so $j^* = \frac{TA^{-\theta}}{TA^{-\theta}+\widetilde{T}}$.

Interpreting A as a function of j, we can invert the expression to obtain that,

$1 - j = \frac{\widetilde{T}}{T[A(j)]^{-\theta}+\widetilde{T}} \Rightarrow T\left[A(j)\right]^{-\theta} = \frac{\widetilde{T}}{1-j} - \widetilde{T}$

$\Rightarrow A(j) = \left(\frac{\widetilde{T}}{T}\frac{j}{1-j}\right)^{\frac{-1}{\theta}} \Rightarrow A(j) = \left(\frac{\widetilde{T}}{T}\right)^{\frac{-1}{\theta}} \left(\frac{j}{1-j}\right)^{\frac{-1}{\theta}}$

Given this expression for $A(j)$, the model becomes isomorphic to DFS. This requires adding the trade-balanced condition, which remains unchanged. Furthermore, country (H) produces all goods in the interval $[0, j^*]$, where j^* satisfies $A(j^*) = \frac{w}{\widetilde{w}}$. The only difference is that j^* is now the fraction of goods that (H) is *expected* to produce.

Keep in mind that, when we work with a continuum of goods, the ***expected*** fraction of goods to be produced by (H) coincides with the fraction of goods that are ***actually produced*** by it. This is because, even though there is uncertainty about whether (H) produces a good, given a continuum of goods there is no uncertainty in the aggregate. As we have mentioned, this is an implication of the Law of Large Numbers. For this reason, in this model all the results corresponding to aggregate variables are not stochastic but actually deterministic.

3 The EK Model

Building on the intuitions from the DFS, next we describe the EK model. We first describe setup, and then derive several probabilistic distributions describing the trade patterns.

3.1 Supply Side

There is a discrete set of countries $\mathcal{C} := \{1, 2, ..., C\}$, and a continuum of goods $\Omega := [0, 1]$. Each good $\omega \in \Omega$ is homogeneous and produced under perfect competition. Labor is the only production factor, which we suppose mobile within countries but not between them.[3]

Unlike DFS, countries have a stochastic technology to produce a good, characterized by random unit requirements. The technology exhibits constant returns to scale and is country-specific. Moreover, every firm has access to the same technology, and so it gets the same draw of labor unit requirement. Since there is only one production factor, the unit labor requirements are given by the inverse of marginal productivity. Due to this feature, we can completely characterize the technology by specifying a distribution for one of these concepts. Following EK, but unlike DFS, we do it through marginal productivity, rather than unit labor requirements.

Formally, denote $\underline{z}_i := (\underline{z}_i(\omega))_{\omega \in [0,1]}$ the random vector describing country i's efficiency to produce goods. The random variable $\underline{z}_i(\omega)$ for good ω defines another random variable: the unit labor requirements $\underline{a}_i(\omega)$, such that $\underline{a}_i(\omega) = \frac{1}{\underline{z}_i(\omega)}$. The relation between both concepts arises since since there is only one production factor. A specific draw of productivity for good ω is denoted $z_i(\omega)$. so that producing one unit of ω requires $a_i(\omega) = \frac{1}{z_i(\omega)}$ units of labor.

We suppose that each $\underline{z}_i(\omega)$ is drawn independently for each ω. The cdf of $\underline{z}_i(\omega)$ is denoted by F_i, which represents i's fraction of goods that have efficiency lower than a certain value. The distribution of $\underline{z}_i(\omega)$ is supposed to be Fréchet:

$$F_i(z_i(\omega)) := \exp\left(-T_i(z_i(\omega))^{-\theta}\right),$$

where $\theta > 1$. Notice that we are implicitly assuming that the distribution is not good-specific (neither T_i nor θ depend on ω). Consequently, the distribution of efficiency for each good is the same within a country.[4]

[3] EK relax the assumption of input mobility between countries by considering the existence of intermediate inputs. These goods can be either consumed or used to produce another good. Since any good is tradable, the assumption implies that the intermediate inputs are mobile between countries. In addition, keep in mind that we are considering only the case of one production factor. The model can be generalized to multiple factors by assuming constant returns to scale technologies. This is also true in DFS.

[4] Notice that the random variable representing efficiency is the same for all goods, but, ex post,

There are several distributions widely used in the Trade literature. One of them is the Fréchet distribution. Its main property is that the maximum of n random variables distributed Fréchet is also Fréchet. This will be relevant for the determination of prices paid by country for a product, identified by the cheapest source country.

The random variable describing the unitary cost of a good ω in country i is $\underline{c}_i(\omega)$. Given draws of productivity in each country, the cost of producing one unit of ω in country i is:

$$c_i(\omega) = a_i(\omega) w_i = \frac{w_i}{z_i(\omega)}.$$

The random price of a firm from i in j is given by

$$\underline{p}_{ij}(\omega) = \frac{w_i}{\underline{z}_i(\omega)} \tau_{ij}. \tag{1}$$

Given realizations of efficiencies in each country, we can identify the price $p_{ij}(\omega)$ that each country i sets in j. This in turn determines the price of the good ω in country j:

$$p_j(\omega) := \inf \{ p_{ij}(\omega) : i \in \mathcal{C} \}.$$

Finally, delivering one unit of ω to country j when the good is produced in $i \neq j$ is subject to an iceberg trade cost τ_{ij}. Consequently, τ_{ij} units need to be sent from i to j in order to have one unit arriving in j. We adopt the convention that there are no costs in the domestic market, so that $\tau_{ii} = 1$, and that trade costs are symmetric, so that $\tau_{ij} = \tau_{ji}$ with $\tau_{ij} > 1$.

3.2 Demand Side

Preferences are identical in each country and given by a symmetric CES utility function:

$$U\left[(q_\omega)_{\omega \in [0,1]}\right] := \left[\int_0^1 (q_\omega)^{\frac{\sigma-1}{\sigma}} \mathrm{d}\omega\right]^{\frac{\sigma}{\sigma-1}},$$

where $\sigma > 1$ and is referred to as the elasticity of substitution.

Later in the course, we will elaborate on the properties of the CES utility. The demand side in EK is unimportant, as its only relevance comes from pinning down the equilibrium quantities and welfare. Instead, the main implications in terms of production and trade patterns are entirely driven by the supply side. For our purposes, there are

different sectors are going to get different draws.

only two features of the CES that you need to know for EK: the utility is symmetric, and the indirect utility through which we measure welfare is given by real income.

Symmetry implies that one unit of any good provides the same level of utility. Hence, if the prices of two goods are equal, the demand and expenditure of all goods are equal too. As for welfare, the CES measures it through real wages, which resembles the case of Cobb Douglas preferences.[5] Denoting the indirect utility for country i by V_i,

$$V_i\left(w_i, \mathbb{P}_i\right) := \frac{Y_i}{\mathbb{P}_i},$$

where Y_i is total income (which will equal wages since there will be no profits in equilibrium) and $\mathbb{P}_i$ is a price index given by:

$$\mathbb{P}_l := \left[\int_0^1 \left[p\left(\omega\right)\right]^{1-\sigma} \mathrm{d}\omega\right]^{\frac{1}{1-\sigma}}.$$

Remember that $\mathbb{P}_i$ is the price of one representative bundle consumed by an agent. This basket is referred to as a quantity index, and is defined such that one quantity index gives one unit of utility.

3.3 Distributions

Before analyzing the model and its implications, let's obtain a few distributions. They are necessary for some of the calculations performed later.

First, let's determine the distribution of $\underline{p}_{ij}\left(\omega\right)$, defined by (1). The term $G_{ij}\left(p;\omega\right) := \Pr\left[\underline{p}_{ij}\left(\omega\right) \leq p\right]$ indicates the probability that the good ω produced in country i has a price lower than p when it is delivered to j. Given the assumptions of the model, this takes the following form:

$$G_{ij}\left(p;\omega\right) = 1 - \exp\left[-T_i p^{\theta}\left(w_i \tau_{ij}\right)^{-\theta}\right].$$

We want to get the expression for $G_{ij}\left(p;\omega\right) := \Pr\left[\underline{p}_{ij}\left(\omega\right) \leq p\right]$. Since $\underline{p}_{ij}\left(\omega\right) = \frac{w_i}{\underline{z}_i(\omega)}\tau_{ij}$, then
$G_{ij}\left(p;\omega\right) := \Pr\left[\underline{p}_{ij}\left(\omega\right) \leq p\right] \Rightarrow G_{ij}\left(p;\omega\right) = \Pr\left[\frac{w_i}{\underline{z}_i(\omega)}\tau_{ij} \leq p\right]$
$\Rightarrow G_{ij}\left(p;\omega\right) = \Pr\left[\frac{w_i}{p}\tau_{ij} \leq \underline{z}_i\left(\omega\right)\right] \Rightarrow \Pr\left[\underline{z}_i\left(\omega\right) \leq \frac{w_i}{p}\tau_{ij}\right] = 1 - G_{ij}\left(p;\omega\right)$

[5] This is not a coincidence. Both utility functions are homothetic, and any homothetic utility function has an indirect utility function that can be represented as real income. Furthermore, the Cobb Douglas utility is a particular case of the CES that arises when $\sigma \to 1$.

This implies that $F_i\left(\frac{w_i}{p}\tau_{ij}\right) = 1 - G_{ij}(p;\omega)$ so that $G_{ij}(p;\omega) = 1 - F_i\left(\frac{w_i}{p}\tau_{ij}\right)$.
Since $F_i(z_i(\omega)) := \exp\left(-T_i[z_i(\omega)]^{-\theta}\right)$, then $F_i\left(\frac{w_i}{p}\tau_{ij}\right) = \exp\left[-T_i p^{\theta}(w_i\tau_{ij})^{-\theta}\right]$ and so $G_{ij}(p;\omega) = 1 - \exp\left[-T_i p^{\theta}(w_i\tau_{ij})^{-\theta}\right]$.

Now, let's determine the distribution of the random variable representing the price of good ω in j. Since goods are homogeneous, the price that consumers in j will pay for good ω is given by the minimum price:

$$\underline{p}_j(\omega) := \inf_{c\in\mathcal{C}}\left\{\underline{p}_{cj}(\omega)\right\}.$$

Respectively denoting the cdf and pdf of this variable by $G_j(p;\omega)$ and $g_j(p;\omega)$,

$$G_j(p;\omega) = 1 - \exp\left[-\Phi_j p^{\theta}\right],$$
$$g_j(p;\omega) = \exp\left[-\Phi_j p^{\theta}\right]\Phi_j\theta p^{\theta-1},$$

where $\Phi_j := \sum_{c\in\mathcal{C}} T_c(w_c\tau_{cj})^{-\theta}$. Notice that the distribution G_j is the same for any good ω and is distributed Fréchet.

By definition $G_j(p;\omega) = \Pr\left[\inf_{c\in\mathcal{C}} \underline{p}_{cj}(\omega) \leq p\right]$ and refers to the event comprising the states where, given draws of efficiency in each country, at least one country sets the price p and any other country sets a price equal or higher than p.
For the calculations, it is easy to note that this event is the complement of the event in which all countries are setting a price equal or higher than p. Thus, given the independence of distributions, we have that:
$G_j(p;\omega) = 1 - \Pr\left[\underline{p}_j(\omega) \geq p\right] \Rightarrow G_j(p;\omega) = \Pr\left[\bigcap_{c\in\mathcal{C}}\left\{\underline{p}_{cj}(\omega) \geq p\right\}\right]$
$\Rightarrow G_j(p;\omega) = 1 - \prod_{c\in\mathcal{C}}\left[1 - G_{cj}(p;\omega)\right]$
$\Rightarrow G_j(p;\omega) = 1 - \prod_{c\in\mathcal{C}}\exp\left[-T_c p^{\theta}(w_c\tau_{cj})^{-\theta}\right]$
$\Rightarrow G_j(p;\omega) = 1 - \exp\left[\sum_{c\in\mathcal{C}} -T_c p^{\theta}(w_c\tau_{cj})^{-\theta}\right]$
and defining $\Phi_j := \sum_{c\in\mathcal{C}} T_c(w_c\tau_{cj})^{-\theta}$ then $G_j(p;\omega) = 1 - \exp\left[\Phi_j p^{\theta}\right]$ which defines the density $g_j(p;\omega) = \frac{dG_j(p;\omega)}{dp} = \exp\left[-\Phi_j p^{\theta}\right]\Phi_j\theta p^{\theta-1}$.

Next, we calculate the probability that the random price of a good ω produced in i results in the minimum price in country j. Formally, this is $\pi_{ij} := \Pr\left[\underline{p}_{ij}(\omega) \leq \inf_{c\in\mathcal{C}\setminus\{i\}} \underline{p}_{cj}(\omega)\right]$, and provides information about the likelihood that country i serves country j. It is given by

$$\pi_{ij} = \frac{T_i(w_i\tau_{ij})^{-\theta}}{\Phi_j} = \frac{T_i(w_i\tau_{ij})^{-\theta}}{\sum_{c\in\mathcal{C}} T_c(w_c\tau_{cj})^{-\theta}}. \tag{2}$$

Calculating $\pi_{ij} := \Pr\left[\underline{p}_{ij}(\omega) \leq \inf_{c\in\mathcal{C}\backslash\{i\}} \underline{p}_{cj}(\omega)\right]$ requires considering all the possible realizations of draws that the country i can have, since $\underline{p}_{ij}(\omega)$ is a random variable. In addition, we also need to consider the draws that all the other countries could get, since each $\underline{p}_{cj}(\omega)$ is a random variable too.

To calculate π_{ij}, we proceed as follows. We start by considering a realization p of $\underline{p}_{ij}(\omega)$. The probability that p is the lowest price is the probability that the prices of all the other countries are greater than p. Formally, this is the probability that $\underline{p}_{cj} \geq p$ for each $c \neq i$. After this, we consider all the possible values p that country i could get as a draw. Formally,

$\pi_{ij} := \Pr\left[\underline{p}_{ij}(\omega) \leq \inf_{c\in\mathcal{C}\backslash\{i\}} \underline{p}_{cj}(\omega)\right]$

$\Rightarrow \pi_{ij} = \int_{p\in[0,\infty)} \Pr\left[p \leq \inf_{c\in\mathcal{C}\backslash\{i\}} \underline{p}_{cj}(\omega)\right] \mathrm{d}G_{ij} \Rightarrow \pi_{ij} := \int_0^\infty \Pr\left[\bigcap_{c\in\mathcal{C}\backslash\{i\}} \left\{p \leq \underline{p}_{cj}(\omega)\right\}\right] \mathrm{d}G_{ij}$

$\Rightarrow \pi_{ij} = \int_0^\infty \prod_{c\neq i} \Pr\left[p \leq \underline{p}_{cj}(\omega)\right] \mathrm{d}G_{ij} \Rightarrow \pi_{ij} := \int_0^\infty \prod_{c\neq i} \left[1 - G_{cj}(p;\omega)\right] \mathrm{d}G_{ij}$

$\Rightarrow \pi_{ij} = \int_0^\infty \prod_{c\neq i} \exp\left[-T_c p^\theta (w_c\tau_{cj})^{-\theta}\right] \mathrm{d}G_{ij} \Rightarrow \pi_{ij} := \int_0^\infty \exp\left[-p^\theta \sum_{c\neq i} T_c (w_c\tau_{cj})^{-\theta}\right] \mathrm{d}G_{ij}$.

Knowing that the density function g_{ij} is given by $g_{ij}(\omega;p) := \frac{\mathrm{d}G_{ij}(p;\omega)}{\mathrm{d}p}$:

$g_{ij}(\omega;p) := \exp\left[-T_i p^\theta (w_i\tau_{ij})^{-\theta}\right] T_i (w_i\tau_{ij})^{-\theta} \theta p^{\theta-1}$

Then,

$\pi_{ij} = \int_0^\infty \exp\left[-p^\theta \sum_{c\neq i} T_c (w_c\tau_{cj})^{-\theta}\right] \mathrm{d}G_{ij}$

$\Rightarrow \pi_{ij} = \int_0^\infty \exp\left[-p^\theta \sum_{c\neq i} T_c (w_c\tau_{cj})^{-\theta}\right] \exp\left[-T_i p^\theta (w_i\tau_{ij})^{-\theta}\right] T_i (w_i\tau_{ij})^{-\theta} \theta p^{\theta-1} \,\mathrm{d}p$

$\Rightarrow \pi_{ij} = \theta T_i (w_i\tau_{ij})^{-\theta} \int_0^\infty \exp\left[-p^\theta \sum_{c\in\mathcal{C}} T_c (w_c\tau_{cj})^{-\theta}\right] p^{\theta-1} \,\mathrm{d}p$

$\Rightarrow \pi_{ij} = \theta T_i (w_i\tau_{ij})^{-\theta} \int_0^\infty \exp\left[-p^\theta \Phi_j\right] p^{\theta-1} \,\mathrm{d}p$

and we can use that $g_j(p;\omega) = \exp\left[-\Phi_j p^\theta\right] \Phi_j \theta p^{\theta-1}$ so that, multiplying and dividing by Φ_j:

$\pi_{ij} = \frac{T_i(w_i\tau_{ij})^{-\theta}}{\Phi_j} \underbrace{\int_0^\infty \theta\Phi_j \exp\left[-p^\theta \Phi_j\right] p^{\theta-1} \,\mathrm{d}p}_{=1}$

Finally, let's determine the distribution of the price paid by j for a good produced in i, when country i is the cheapest source in j. Formally, this is

$$H_{ij}(p) := \Pr\left[\underline{p}_{ij}(\omega) \leq p \,\middle|\, \underline{p}_{ij}(\omega) \leq \inf_{c\in\mathcal{C}\backslash\{i\}} \underline{p}_{cj}(\omega)\right],$$

and it can be shown that

$$H_{ij}(p) = G_j(p). \tag{3}$$

Let's show that $H_{ij}(p) = G_j(p)$. By definition $H_{ij}(p) := \Pr\left[\underline{p}_{ij}(\omega) \leq p \,\middle|\, \underline{p}_{ij}(\omega) \leq \inf_{c\in\mathcal{C}\backslash\{i\}} \underline{p}_{cj}(\omega)\right]$ so that $H_{ij}(p) := \frac{\int_0^p \Pr\left[\bigcap_{c\in\mathcal{C}\backslash\{i\}} \left\{q \leq \underline{p}_{cj}\right\}\right] g_{ij}(q)\,\mathrm{d}q}{\pi_{ij}}$.

We know that $g_{ij}(\omega;p) := \exp\left[-T_i p^\theta (w_i\tau_{ij})^{-\theta}\right] T_i (w_i\tau_{ij})^{-\theta} \theta p^{\theta-1}$ and also $\Pr\left[\bigcap_{c\in\mathcal{C}\backslash\{i\}} \left\{q \leq \underline{p}_{cj}\right\}\right] = \prod_{c\neq i} \left[1 - G_{cj}(q;\omega)\right]$ which gives $\Pr\left[\bigcap_{c\in\mathcal{C}\backslash\{i\}} \left\{q \leq \underline{p}_{cj}\right\}\right] = \exp\left[-q^\theta \sum_{c\neq i} T_c (w_c\tau_{cj})^{-\theta}\right]$. Hence,

$H_{ij}(p) = \frac{1}{\pi_{ij}} \int_0^p \exp\left[-q^\theta \sum_{c\neq i} T_c (w_c\tau_{cj})^{-\theta}\right] \exp\left[-T_i q^\theta (w_i\tau_{ij})^{-\theta}\right] T_i (w_i\tau_{ij})^{-\theta} \theta q^{\theta-1} \,\mathrm{d}q$

$\Rightarrow H_{ij}(p) = \frac{1}{\pi_{ij}} \int_0^p \exp\left[-q^\theta \sum_{c\in\mathcal{C}} T_c (w_c\tau_{cj})^{-\theta}\right] T_i (w_i\tau_{ij})^{-\theta} \theta q^{\theta-1} \,\mathrm{d}q$

and by multiplying and dividing by Φ_j:

$\Rightarrow H_{ij}(p) = \frac{1}{\pi_{ij}} \underbrace{\frac{T_i (w_i\tau_{ij})^{-\theta}}{\sum_{c\in\mathcal{C}} T_c (w_c\tau_{cj})^{-\theta}}}_{=\pi_{ij}} \underbrace{\int_0^p \exp\left[-q^\theta \sum_{c\in\mathcal{C}} T_c (w_c\tau_{cj})^{-\theta}\right] \sum_{c\in\mathcal{C}} T_c (w_c\tau_{cj})^{-\theta} \theta q^{\theta-1} \,\mathrm{d}q}_{=G_j(p)}$

Equation (3) has several implications for the conclusions of the model. Each i could have a different parameter T_i, and so different absolute advantages. However, conditional on serving the market j, any country would charge the same price in country j, on average. This follows because $H_{ij}(p)$ does not depend on i, since it equals $G_j(p)$ for any $i \in \mathcal{C}$. In other terms, the origin of the good is irrelevant for the price determination in expectation, and thus conditioning on the source has no bearing on the determination of a good's price.

Additionally, the distribution of prices charged by each country in j is the same. This implies that, since utility is symmetric, the distribution of quantities sold in j by each country i is also the same. Thus, each country has the same distribution of prices and of quantities sold in j (both conditional on serving the market). This entails that, conditional on serving j, the expenditure on each good by any consumer from j is the same in expectations. As a consequence, the total expenditure share of country j in goods from i coincides with the fraction of goods produced in i and sold in j. Formally,

$$\frac{E_{ij}}{E_j} = \pi_{ij},$$

where E_{ij} is the expenditure of country j on goods produced by i and E_j is the total expenditure of country j.

4 Taking the EK Model to the Data

We have observed that one significant contribution of EK is its ability to connect the model with real-world data. In Economics jargon, when a model enables us to do this we say it is a structural model. Basically, this means that we can estimate some of the model parameters, and then use the model to quantify the impact of shocks in the economy.

Specifically, EK can be used to get predictions about how changes in trade costs affect:

[1] Welfare

[2] Bilateral trade

To set the model in a way we can take it to the data, it is necessary to rewrite some parts of the model. In particular, this entails finding a set of equations that are functions of observables. In this way, we can compute the results of the model given some data.

Next, we show how different equations can be rewritten for this purpose.

4.1 Welfare

We use the indirect utility function to measure welfare. To do this, we begin by getting an expression for the price index. For country i, and assuming $1+\theta-\sigma>0$, this is given by

$$\mathbb{P}_i = \left[\Gamma\left(\frac{1-\sigma+\theta}{\theta}\right)\right]^{\frac{1}{1-\sigma}} (\Phi_i)^{-\frac{1}{\theta}},$$

where Γ is the Gamma function.[6]

The price index of a CES is given by $(\mathbb{P}_i)^{1-\sigma} = \int_0^1 [p_i(\omega)]^{1-\sigma}\, \mathrm{d}\omega$. We use the distribution of prices to perform the integral. We know that the price distribution has cdf $G_i(p;\omega) = 1-\exp\left(-\Phi_i p^{\theta}\right)$ and pdf $g_i(p;\omega) = \exp\left(-\Phi_i p^{\theta}\right)\Phi_i \theta p^{\theta-1}$. So

$(\mathbb{P}_i)^{1-\sigma} = \int_0^1 [p_i(\omega)]^{1-\sigma}\, \mathrm{d}\omega = \int_p t^{1-\sigma} g_i(t)\, \mathrm{d}t$

$\Rightarrow (\mathbb{P}_i)^{1-\sigma} = \int_p t^{1-\sigma} \exp\left(-\Phi_i t^{\theta}\right)\Phi_i \theta t^{\theta-1}\, \mathrm{d}t$.

We express the results in terms of the Gamma function, defined by $\Gamma(z) := \int_0^\infty x^{z-1}\exp(x)\, \mathrm{d}x$. To do this, let $u := \Phi_i t^{\theta}$ so that $du = \Phi_i \theta t^{\theta-1}\mathrm{d}t$. Notice that $t = \left(\frac{u}{\Phi_i}\right)^{\frac{1}{\theta}}$. Then,

$(\mathbb{P}_i)^{1-\sigma} = \int_p \left(\frac{u}{\Phi_i}\right)^{\frac{1-\sigma}{\theta}} \exp(-u)\, \mathrm{d}u$

$\Rightarrow (\mathbb{P}_i)^{1-\sigma} = \left(\frac{1}{\Phi_i}\right)^{\frac{1-\sigma}{\theta}} \int_p u^{\left(\frac{1-\sigma}{\theta}+1\right)-1} \exp(-u)\, \mathrm{d}u$

and assuming that $1+\theta-\sigma>0$, then

$\Rightarrow (\mathbb{P}_i)^{1-\sigma} = \left(\frac{1}{\Phi_i}\right)^{\frac{1-\sigma}{\theta}} \Gamma\left(\frac{1+\theta-\sigma}{\theta}\right)$

$\Rightarrow \mathbb{P}_i = (\Phi_i)^{\frac{-1}{\theta}} \left[\Gamma\left(\frac{1+\theta-\sigma}{\theta}\right)\right]^{\frac{1}{1-\sigma}}$

In last instance, we do not care about the value of $\mathbb{P}_i$ itself, but rather its change when two different scenarios are considered. Generally speaking, when we have multiplicative terms, it is easier to express the change of a variable in percentage terms. Since the first term of the right-hand side of $\mathbb{P}_i$ plays no role (it acts as a constant), it is convenient to define $\gamma := \left[\Gamma\left(\frac{1-\sigma+\theta}{\theta}\right)\right]^{\frac{1}{1-\sigma}}$ and reexpress the price index as:

$$\mathbb{P}_i = \gamma (\Phi_i)^{-\frac{1}{\theta}}.$$

[6] The Gamma function is given by $\Gamma(z) := \int_0^\infty x^{z-1}\exp(x)\, \mathrm{d}x$. It is a generalization of factorials to real numbers, so that if z is an integer, then $\Gamma(z) = (z-1)!$. More generally, if z is a real number, Γ satisfies that $\Gamma(z+1) = z\Gamma(z)$.

Likewise, the indirect utility function in country i is $\frac{w_i}{\mathbb{P}_i}$, which can be expressed by

$$\frac{w_i}{\mathbb{P}_i} = \gamma^{-1} (T_i)^{\frac{1}{\theta}} (\pi_{ii})^{\frac{-1}{\theta}} . \tag{4}$$

To get the expression, we start from $\pi_{ii} = \frac{T_i w_i^{-\theta}}{\Phi_i}$. Using that $\left(\frac{\mathbb{P}_i}{\gamma}\right)^{-\theta} = \Phi_i$, then $\pi_{ii} = \frac{T_i (w_i)^{-\theta}}{\left(\frac{\mathbb{P}_i}{\gamma}\right)^{-\theta}}$ which implies that $\pi_{ii} = \gamma^{-\theta} T_i \left(\frac{w_i}{\mathbb{P}_i}\right)^{-\theta}$ and the result follows.

In autarky, we know that $\pi_{ii}^{aut} := 1$. Therefore, we can use the indirect utility expressed in the form (4) to get the change in welfare relative to autarky:

$$\frac{w_i/\mathbb{P}_i}{(w_i/\mathbb{P}_i)^{aut}} = (\pi_{ii})^{-1/\theta} .$$

Notice π_{ii} can be easily obtained from the official statistics collected by countries. Thus, given an estimation of θ, we can estimate the welfare gains of the current situation relative to autarky.

4.2 Bilateral Trade: The Gravity Equation

The bilateral trade predicted by the model obeys the gravity equation:

$$E_{ij} = \frac{(\mathbb{P}_j)^{\theta}}{\sum_{c \in \mathcal{C}} \left(\frac{\mathbb{P}_c}{\tau_{ic}}\right) E_c} (\tau_{ij})^{-\theta} E_j Y_i. \tag{5}$$

We start from the equation $\frac{E_{ij}}{E_j} = \pi_{ij} = \frac{T_i (w_i \tau_{ij})^{-\theta}}{\Phi_j}$ so that $E_{ij} = \frac{T_i (w_i \tau_{ij})^{-\theta}}{\Phi_j} E_j$.
Using that $Y_i = \sum_{c \in \mathcal{C}} E_{ic}$, then
$Y_i = \sum_{c \in \mathcal{C}} \frac{T_i (w_i \tau_{ic})^{-\theta}}{\Phi_c} E_c \Rightarrow \frac{Y_i}{\sum_{c \in \mathcal{C}} \frac{(\tau_{ic})^{-\theta} E_c}{\Phi_c}} = T_i (w_i)^{-\theta}$.
We plug in the expression for $T_i (w_i)^{-\theta}$ in the equation for E_{ij}, so that
$E_{ij} = T_i (w_i)^{-\theta} \frac{(\tau_{ij})^{-\theta} E_j}{\Phi_j} \Rightarrow E_{ij} = \frac{Y_i}{\sum_{c \in \mathcal{C}} \frac{(\tau_{ic})^{-\theta} E_c}{\Phi_c}} \frac{(\tau_{ij})^{-\theta} E_j}{\Phi_j}$.
Finally, the expression for the price index we found above determine that $\left(\frac{\mathbb{P}_c}{\gamma}\right)^{-\theta} = \Phi_c$, and so
$E_{ij} = \frac{Y_i}{\sum_{c \in \mathcal{C}} \frac{(\tau_{ic})^{-\theta} E_c}{\left(\frac{\mathbb{P}_c}{\gamma}\right)^{-\theta}}} \frac{(\tau_{ij})^{-\theta} E_j}{\left(\frac{\mathbb{P}_j}{\gamma}\right)^{-\theta}}$ which reordering it becomes $\Rightarrow E_{ij} = Y_i E_j \frac{\left(\frac{\mathbb{P}_j}{\tau_{ij}}\right)^{\theta}}{\sum_{c \in \mathcal{C}} \left(\frac{\mathbb{P}_c}{\tau_{ic}}\right)^{\theta} E_c}$.

5 Computation and Counterfactuals

For welfare calculations, we can proceed without identifying all the equilibrium values. Given an estimation of θ, the term π_{ii} can be obtained through data. Thus, all the terms in (4) can be easily computed.

Similarly, the gravity equation might be used without need to determine the equilibrium. For instance, EK use the log version of (5) to provide an estimation of θ. By parametrizing trade costs and treating the rest of the variables as fixed effects, they do not need to solve for the equilibrium.

However, if our goal is to use the model to compare an observed situation relative to an unobserved counterfactual, we need to solve for the equilibrium of the model. This requires establishing the remaining equilibrium conditions. With this goal, we use that each country's income has to equal its expenditure:

$$w_i L_i = E_i.$$

The total income of country i is given by $w_i L_i$ and determined by the total revenue generated in the economy. Total revenue equals the value of exports and domestic sales, which are given by $\sum_{j\in\mathcal{C}} E_{ij}$. Since $E_{ij} = \pi_{ij} E_j$, we get that:

$$w_i L_i = \sum_{j\in\mathcal{C}} \pi_{ij} E_j.$$

Thus, wages can be solved by using the following system of equations:

$$w_i L_i = \frac{\sum_{j\in\mathcal{C}} T_i \left(w_i \tau_{ij}\right)^{-\theta} w_j L_j}{\sum_{c\in\mathcal{C}} T_c \left(w_c \tau_{cj}\right)^{-\theta}}. \tag{6}$$

The equation (6) follows by using that:

$w_i L_i = \sum_{j\in\mathcal{C}} \pi_{ij} E_j$

$\Rightarrow w_i L_i = \sum_{j\in\mathcal{C}} \pi_{ij} w_j L_j$

because $E_j = w_j L_j$, and then using that $\pi_{ij} = \frac{T_i\left(w_i \tau_{ij}\right)^{-\theta}}{\sum_{c\in\mathcal{C}} T_c\left(w_c \tau_{cj}\right)^{-\theta}}$.

Once that wages for each country are obtained, variables such as trade flows and price indices can be obtained. Notice that using the model empirically supposes we have estimated or assigned values to $(T_i, L_i)_{i\in\mathcal{C}}$, θ, σ and $(\tau_{ij})_{i,j\in\mathcal{C}}$.